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Oracles

$Max\,SAT = \{ F, k$ s.t. $\exists \alpha$ satisfying k clauses of F and $\forall \beta$ satisfies $\leq k$ clauses of $F \}$.

prop $Max\,SAT \in P^{SAT}$.

$F = C_1 \wedge C_2 \wedge \dots C_m$ original formula.

Let $G_\ell = (C_1 \vee r_1) \wedge (C_2 \vee r_2) \wedge \dots \wedge (C_m \vee r_m)$

$\wedge [r_1 + r_2 + \dots r_m \leq m - \ell]$

↳ encode using adder circuit plus comparison circuit.

Claim: $G_\ell \in SAT$ iff F has assignment satisfying $\geq \ell$ clauses.

proof: α sat ℓ clauses in $F \Rightarrow \beta = \alpha$ extended to $r_i = [\alpha$ satisfies $C_i]$ has $\leq m-\ell$ r-variables set to 1

$\Rightarrow G_\ell$ satisfied.

β satisfies $G_\ell \Rightarrow \beta$ has $\leq m-\ell$ r-variables set to 1

$\Rightarrow \alpha = \beta$ restricted satisfies $\geq \ell$ clauses in F.

Hence the following oracle machine solves Max SAT in poly time:

M^{SAT}: input (F, k).

if $G_k \in SAT$ and $G_{k+1} \notin SAT$:
accept
o/w:
reject

Cook–Levin theorem.

th SAT is NP-complete.

proof. Let $L \in NP$, i.e. $L = \{x : \exists^p y, V(x,y)$ accepts$\}$.

Want to show $L \leq_p SAT$. We build a formula that simulates the computation of V.

* note it is the same formula we used in Gödel's incompleteness thm.

We'll encode that $s_0, s_1, \dots, s_m$ is a valid computation. Have variables z_{ijk} representing state i/symbol j/bit k.

Encode: $s_0 = q_0 x y$.

$s_{i,j-1}$ $s_{i,j}$ $s_{i,j+1}$

$s_{i+1,j-1}$ $s_{i+1,j}$ $s_{i+1,j+1}$

→ If $s_{i,j-1} = q_a$ and $\delta(q_a, s_{i,j}) = (q_b, c, R)$

then $s_{i+1,j-1} = c$ and $s_{i+1,j} = q_b$

→ If $s_{i,j} = q_a$ and $\delta(q_b, s_{i,j+1}) = (q_b, c, L)$

then $s_{i+1,j+1} = c$ and $s_{i+1,j-1} = q_b$

→ O/w $s_{i+1,j} = s_{i,j}$

Obs we are encoding a condition over constant many variables. Hence exists CNF of constant size.

We need $|s_i| = poly(n)$ checks per state and we have $m = poly(n)$ many states. Hence total size of CNF is $poly(n)$.

This proves reduction is poly size. Correctness is easy to check from definitions.