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Recall:

$L \in \Sigma^1$ if $L = \{x : \exists y \ \psi(x,y)\}$ where ψ is decidable.

$L \in NP$ if $L = \{x : \exists^P y \ V(x,y) \text{ accepts}\}$ where V is a poly-time TM.

Computability	→	(Time) Complexity
Decidable	→	Poly-time (P)
$\exists x$	→	$\exists^P y$ ($\exists y, \lvert y \rvert = poly$)
Recognizable (Σ^1)	→	NP ($\Sigma^1 P$)
co-recognizable (Π^1)	→	co-NP ($\Pi^1 P$)
UNI	→	SAT
$\Sigma^i / \Pi^i / \Delta^i$	→	$\Sigma^i P / \Pi^i P / \Delta^i P$
Arithmetic Hierarchy		Polynomial Hierarchy

More formally, $L \in \Sigma^i P$ if $\exists V$ st

$x \in L$ iff $\exists y_1 \forall y_2 \exists y_3 \cdots Q y_i \ V(x, y_1, \ldots, y_i)$.

Examples.

e.g. $\Sigma^1 P = NP$. $SAT \in \Sigma^1 P$.

e.g. $\Pi^1 P = coNP$. $\overline{SAT} \in \Pi^1 P$.

e.g. Max SAT: find assignment to vars that satisfies the most clauses. As a decision problem:

$(F, k) \in$ Max SAT if max satisfiable clauses $= k$.

iow: $\exists \alpha$ st α sat. k clauses and $\forall \beta$, β sat $\leq k$ clauses.

Max SAT $\in \Sigma^2 P$.

e.g. Graph SAT: given formula without negations, is it satisfiable no matter how we choose negations?

iow: $\forall$ negations $\exists$ satisfying assignment.

Graph SAT $\in \Pi^2 P$.

e.g. VC dimension. $\mathcal{H}$ set of sets. C set.

def. $\mathcal{H} \cap C = \{H \cap C : H \in \mathcal{H}\}$.
def. $\mathcal{H}$ shatters C if $\mathcal{H} \cap C = \mathcal{P}(C)$.
def. VC dim of $\mathcal{H} = \max_{\mathcal{H} \text{ shatters } C} |C|$.

$\{\mathcal{H}, k : \text{VC-dim of } \mathcal{H} \geq k\} \in \Sigma^3 P$.

* VC-dim is a useful concept in learning theory.

e.g. Travelling Salesperson (TSP): given weighted graph, find a path visiting every vertex exactly once and minimizing sum of weights in path.

aka Hamiltonian path

As a decision problem:

$(G, k) \in TSP$ if min weights $\leq k$.

iow: $\exists$ path st $\sum_{e \in path} w(e) \leq k$ and path is Hamiltonian.

$TSP \in \Sigma^1 P$.

Alternate formulation:

$(G, k) \in TSP'$ if min weights $= k$.

iow: $\exists$ path st $\sum_{e \in path} w(e) = k$ and path is Hamiltonian
and $\forall$ path, path is Hamiltonian $\Rightarrow \sum_{e \in path} w(e) \geq k$.

$TSP' \in \Sigma^2 P$ and $TSP' \in \Pi^2 P$.

Equivalent results with oracles also hold.

e.g. P^{SAT} are languages decidable by a det. TM with access to SAT oracle, in poly time.

P^{SAT} more powerful than NP (e.g. can solve Max SAT).

NP^{SAT} are languages decidable by a nondet. TM with access to SAT oracle, in poly time.

$NP^{SAT} = \Sigma^2 P$.

* SAT oracles exist in the real world; there are algorithms that are efficient in many practically relevant cases (but not in the worst case).