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Subset Sum is NP-complete.

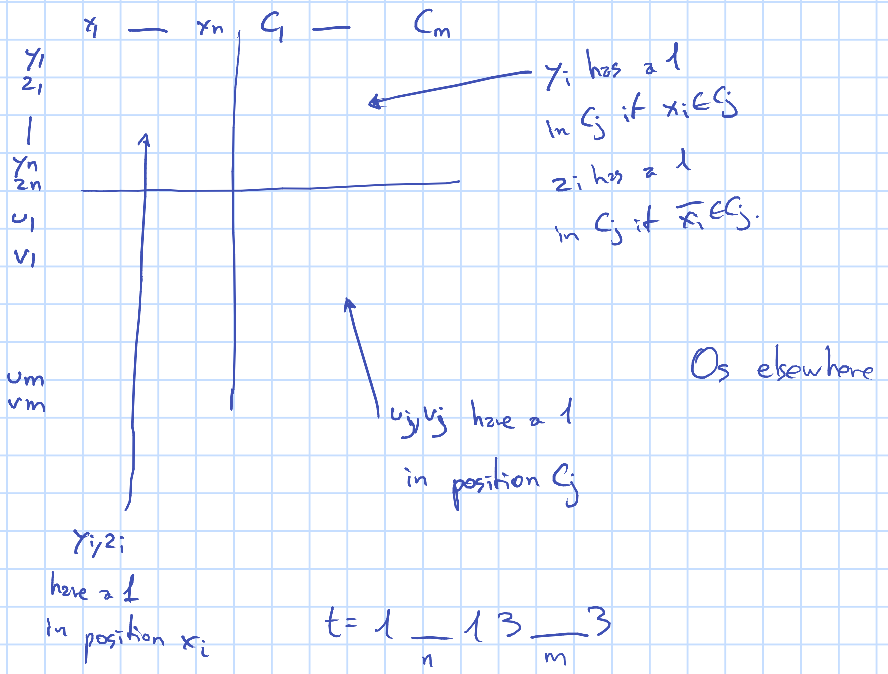
Subset Sum is the following problem:

Given numbers $x_1 - x_n$, is there a subset $S \subseteq [n]$ st $\sum_{i \in S} x_i = t$?

prop $3SAT \leq_p SubsetSum$

Given CNF $F = C_1 \wedge C_2 \wedge \dots \wedge C_m$ over variables $x_1 \dots x_m$, we build instance as follows.

Numbers are $y_1 \dots y_n, z_1 \dots z_n, u_1 \dots u_m, v_1 \dots v_m$; defined as in the following table:



$t = 1\underbrace{\_\_}_{n}1\,3\underbrace{\_\_}_{m}3$

Intuitively, y_i corresponds to x_i, z_i to $\overline{x_i}$, and u_j, v_j are slack for clause C_j.

e.g.

$(x_1 \vee x_2 \vee \overline{x_3})$
$\wedge (x_2 \vee x_3 \vee x_4)$ ->

	x_1	x_2	x_3	x_4	C_1	C_2
y_1	1	0	0	0	1	0
z_1	1	0	0	0	0	0
y_2		1	0	0	1	1
z_2		1	0	0	0	0
y_3			1	0	0	1
z_3			1	0	1	0
y_4				1	0	1
z_4				1	0	0
u_1					1	0
v_1					1	0
u_2						1
v_2						1
t	1	1	1	1	3	3

So problem is choose subset of (100010, 100000, 1011, 10000, 1001, 1010, 101, 100, 10, 10, 1, 1) that sums up to 111133.

Claim: the reduction is poly-time.

Claim: the reduction is correct.

proof: $F \in SAT \Rightarrow f(F) \in SubsetSum$.

Assume α satisfies F. Pick y_i if $\alpha(x_i) = T$, and z_i if $\alpha(x_i) = \bot$. Let s_j be the number of literals in C_j that α satisfies. Cases:

$s_j = 0$: impossible because α satisfies F.
$s_j = 1$: pick u_j and v_j.
$s_j = 2$: pick u_j but not v_j.
$s_j = 3$: do not pick u_j or v_j.

Easy to check picked numbers sum to t.

proof: $f(F) \in SubsetSum \Rightarrow F \in SAT$.

Assume S is a subset summing up to t. Obs there is no carry, so we can treat columns independently. Obs S contains exactly one of y_i and z_i $\forall i$, so we can set $\alpha(x_i) = T$ if $y_i \in S$ and $\alpha(x_i) = \bot$ if $z_i \in S$. def s_j as before. Obs $3 = t_j \leq s_j + 2 \Rightarrow s_j \geq 1$, hence every clause is satisfied.