L24 2022-11-14

Reductions

e.g. $3SAT \leq_p Clique$

Given 3CNF $F = C_1 \land C_2 \land \ldots C_m$ we build (G, k) st G:

-> has 3 nodes per clause, each labelled with a literal

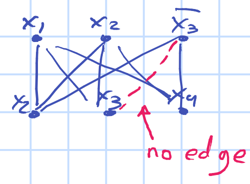
-> has edges between each pair of nodes except:

-> nodes from the same clause

-> nodes with opposite labels.

Formally: have $3m$ nodes; node (i,j) is labelled with the j-th literal of clause C_i. Have edge $(i,j), (i',j')$ if $i \neq i'$ and $label(i,j) \neq \neg label(i',j')$. Set $k = m$.

e.g. $(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor x_4)$ ->



Claim: the reduction is poly-time.

Claim: the reduction is correct.

proof: $F \in SAT \Rightarrow G, k \in Clique$.

Assume α satisfies F. Build a clique as follows: (i,j) is in the clique if it is the first in clause i whose label is satisfied. Because α satisfies ≥ 1 literal per clause, the clique has size m.

proof: $G, k \in Clique \Rightarrow F \in SAT$.

Let $K \subseteq G$ be a clique of size m. Build a truth assignment as follows: $x_i = T$ if a vertex labelled x_i is in the clique, and $x_i = \bot$ o/w. Obs $x_i \in K \Rightarrow \overline{x_i} \notin K$, $\overline{x_i} \in K \Rightarrow x_i \notin K$, so the assignment is well-defined.
Obs K has exactly one vertex per row, hence each clause is satisfied (by the literal corresponding to that vertex).

Hardness & Completeness

def B is $\mathcal{C}$-hard under $\mathcal{F}$-reductions if $\forall A \in \mathcal{C}$ it holds $A \leq_{\mathcal{F}} B$.

Usually $\mathcal{F}$ is clear from context and we omit it.
e.g. RE-hard is under mapping reductions
NP-hard is under poly-time reductions

e.g. UNI is RE-hard

proof: let L be recognizable, M deciding L.

let $f : x \mapsto \langle M, x \rangle$.

f computable & $x \in L$ iff $\langle M, x \rangle \in UNI$.

hence $L \leq_m UNI$.

def L is $\mathcal{C}$-complete if $L \in \mathcal{C}$ and L $\mathcal{C}$-hard.

e.g. UNI is RE-complete.

NP-hardness

th SAT is NP-hard.

proof sketch: a poly-size computation can be verified with a poly-size propositional formula.

We can use reductions to show other problems are NP-hard:

prop: $A \leq_p B$ and A NP-hard then B NP-hard

lemma: $A \leq_p B$ and $B \leq_p C$ then $A \leq_p C$.

proof: $f : A \to B$; $g : B \to C$. Take $g \circ f : A \to C$.
It is poly-time computable because f & g are.
" correct " " .

proof (of prop.) Let $C \in NP$. By def NP-hard, $C \leq_p A$.
By hypothesis, $A \leq_p B$. By lemma, $C \leq_p B$. ※

We showed $SAT \leq_p 3SAT$ and $3SAT \leq_p CLIQUE$, hence they are NP-hard too.

Since SAT, 3SAT, CLIQUE are in NP, they all are NP-complete.

UNI is NP-hard but not in NP, hence not NP-complete.

2SAT is in NP but (probably) not NP-hard, hence (probably) not NP-complete.