L 18 2022-10-31

Self-reference.

Goal: prove Gödel's incompleteness thm.

We'll build a sentence saying

"this sentence is not provable".

(if true not provable; if false then provable, but cannot prove false sentences, hence not false).

Need logical expressions that talk about themselves.

Plan: build logical expressions that talk about TMs, and TMs that talk about themselves.

Today: TMs that talk about themselves (self-referential).

Example of a self-referential problem: a quine.

"print this sentence"

How does a TM say "this TM"?

A high-level program can do it like this:

```
A | s = '...'
B.| print ("s='" + s + "'" + s)
```

where s = 'print ("s='" + s + "'\n" + s)'

A TM can do it in two parts.

lem: the function $w \mapsto \langle P_w \rangle$ where P_w is a TM that prints w is computable.

We'll have $M = A \circ B$, where $A \circ B$ is the TM that moves to B's initial state instead of A's accepting state.

$A = P_{\langle B \rangle}$ (but we don't know B yet).

$B =$ compute P_x
let $N = P_x \circ x$
write $\langle N \rangle$.

$M = A \circ B$ does the following:

1. write $\langle B \rangle$ on tape
2. compute $P_{\langle B \rangle}$ // $P_{\langle B \rangle} = A$ by def
3. let $N = P_{\langle B \rangle} \circ B$ // $N = A \circ B = M$ by def
4. write $\langle N \rangle$

More in general, we can always assume that a TM can access its own description.

th: Let t be computable. Then $\exists\, r$ computable by R st

$$r(x) = t(\langle R \rangle, x).$$

Application: UNI is undecidable.

(alternative) proof.

Assume UNI decidable; let M be a TM for UNI.

Let R be the following TM:

return $\neg M(\langle R \rangle, w)$. allowed because of recursion thm.

Is $\langle R, w \rangle \in \mathrm{UNI}$? by def UNI by def M by def R

if yes, R accepts, but M accepts so R rejects !!.

if no, R does not accept, but M rejects so R accepts !!.

Contradiction $\Rightarrow$ UNI not decidable