L11 2022-10-04

Undecidability.

Can TMs recognize / decide all languages? No!

th: there are more languages than TMs.

lemma: every TM can be represented with a binary string.
(just list its components; c.f. "serialization").

lemma: there is a bijection TMs $\Leftrightarrow \mathbb{N}$.
(list in increasing size order; break ties alphabetically).

lemma: there is no bijection Languages $\Leftrightarrow \mathbb{N}$.
(if you know set theory: Languages $\Leftrightarrow \mathcal{P}(\mathbb{N})$, which has cardinality ω_1, while $\mathbb{N}$ has cardinality ω_0).

this proves th. Let us see proof of lemma.

proof of lemma.

Proof by contradiction. Uses "diagonalization" technique.

Assume there is a bijection languages $\Leftrightarrow \mathbb{N}$.

Write table

$$\begin{array}{cc|cccccc} & & s_1 & s_2 & s_3 & \cdots & s_n & \cdots \text{ strings} \\ \hline & L_1 & a_{11} & a_{12} & a_{13} & & & \\ & L_2 & a_{21} & & & & & \\ & L_3 & a_{23} & & & & & \\ & \vdots & & & & & & \\ \text{languages} & L_n & & & & & & \\ & \vdots & & & & & & \end{array}$$

$a_{ij} = 1$ if set i contains element j
0 o/w.

Consider language $L = \{ s_i \in L \text{ iff } s_{ii} \notin L_i \}$

Its row is $(1-a_{ii})_i$, that is the opposite of the diagonal.

But such a row does not appear in our table!!
(if it were L_m, then $a_{mm} = 1 - a_{mm}$!!).

⋇

Maybe problems that are not decidable are not interesting, so this was not a big deal?

No. Let $UNI = \{ \langle M, s \rangle : M \text{ is a TM that accepts } s \}$.

* notation: $\langle X \rangle$ means the string representation of object X.

th UNI is undecidable.

* note UNI is recognizable, because there exists a TM U st given $\langle M \rangle$ as input, it behaves like M. U is called a universal TM.

proof. Assume for the sake of contradiction a TM H exists that accepts when $\langle M, s \rangle \in UNI$ and rejects o/w.

Let D be the TM that, given $\langle M \rangle$ as input, runs H on input $\langle M, \langle M \rangle \rangle$ and outputs the opposite.

$D(\langle M \rangle)$ rejects if M accepts $\langle M \rangle$
accepts o/w.

What does $D(\langle D \rangle)$ do?

if $D(\langle D \rangle)$ accepts then D does not accept $\langle D \rangle$!!
if $D(\langle D \rangle)$ rejects then D accepts $\langle D \rangle$!!

So $D(\langle D \rangle)$ does not stop. But this contradicts our assumption that H always stops. ⋇

* This proof uses diagonalization in disguise.

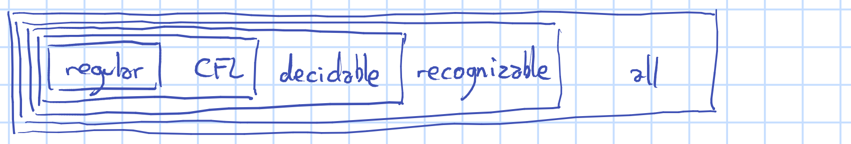
Think of a table

$$\begin{array}{cc|cccc} & & \langle M_1 \rangle & \langle M_2 \rangle & \langle M_3 \rangle & \text{inputs} \\ \hline & M_1 & & & & \\ \text{machines} & M_2 & & & & \\ & \downarrow & & & & \end{array}$$

$a_{ij} = $ "M_i accepts $\langle M_j \rangle$".

D is the negation of the diagonal. If D appeared in the table, its diagonal entry would not make sense.

Obs we can categorize languages:



Obs we still do not know of an explicit not-recognizable language. We build one from UNI: simply $\overline{UNI}$.

To prove that we use the following theorem.

th: L is decidable iff L and $\overline{L}$ recognizable.

Obs follows: UNI is recognizable. If $\overline{UNI}$ were recognizable, then Acc would be decidable.

proof of th:
$\Rightarrow$) If L decidable, then $\overline{L}$ decidable (switch acc / rej).
Hence L and $\overline{L}$ recognizable.

$\Leftarrow$) Assume L and $\overline{L}$ recognizable; let M_1 and M_2 be TMs for L and $\overline{L}$ resp.

Simulate M_1 and M_2 in parallel (with a 2-tape TM; interleaving steps from M_1 and M_2).