EXACT-CLIQUE is DP-complete.

$L = L_1 \cap \overline{L_2}$ $\quad L_1, L_2 \in NP$

$\varphi_1 : x \to (G_1, k)$
$\varphi_2 : x \to (G_2, k)$

k if $x \in L$
$k-1$ o/w.

G has max clique of size exactly k iff $x \in L_1$ and $x \notin L_2$.

~~G = tensor product of G_1, G_2.~~
~~max clique in G = min (— (G_1), — (G_2)).~~

G = strong product of G_1, G_2.

$V(G) = (u_1, u_2) : u_1 \in G_1, u_2 \in G_2$.

$E(G) = ((u_1, u_2), (v_1, v_2))$ iff $(v_1, u_1) \in G_1 \wedge (u_2, v_2) \in G_2$ (with self loops).

if $u_2 = v_2$: always edge.

V_1 clique in G_1
V_2 " G_2
$V_1 \times V_2$ clique in G.

max clique in G = max(G_1) $\cdot$ max (G_2).

$\varphi_1 : x \to (G_1, k)$
$\varphi_2 : x \to (G_2, k+1)$.

	$x \in L_1$	$x \notin L_1$
$x \in L_2$	$k \cdot (k+1)$	$(k-1)(k+1)$
$x \notin L_2$	$k \cdot k$	$(k-1) \cdot k$

k^2

$x \in L$ iff (G, k^2).

D: for each $n \in \mathbb{N}$: with prob $1/2$ no string of length n is in L.
$1/2$ exactly one string of length n in L.
(picked unif. random).

$\Pr[P^L \neq NP^L] = 1$.
$L \sim D$

$B = \{1^n$ if $\exists x \in L, |x| = n\}$.

$B \in NP^L$ ✓.

machines M_i^L $i \in \mathbb{N}$. each M_i asks poly many questions.

$\forall i$: $\Pr[M_i^L$ solves $B] = 0$. want to show $\Pr[M_i(1^n)$ mistake$] > c$

$M_i(1^n)$ asks poly many questions.

if all questions are answered 0: and M_i answers 1 — $M_i(1^n)$ wrong w/ prob $1/2$.

and M_i answers 0
$M_i(1^n)$ wrong w/ prob $1/4$.

$\Pr[M_i(1^n)$ ok$]$
$\Pr[M_i(1^m)$ ok$]$ not independent.

$n_1, n_2, \dots$ $\quad n_{i+1} = 2^{n_i}$ $\quad M_i$ cannot ask strings of length n_{i+1}

could be M^i always asks first k strings.

$\to \prod \Pr[M_i(1^{n_j})$ ok $| M_i(1^{n_{j'}})$ ok $\forall j' < j]$

$\Pr[M_i(1^{n_j})$ ok $| M_i(1^{n_{j'}})$ ok $\forall j' < j] =$

$L = L_1 \cup L_2$

$= \mathbb{E}_{L_1} \Pr_{L_2} [$ — $|$ $\cancel{\quad}$ $]$

L_1 strings of length $< 1^{n_j}$
L_2 $\geq 1^{n_j}$

$\Pr_{L_2}[M_i(1^{n_j})$ ok$] < 3/4$.

$\Pr_{x \sim D} = \sum_y \Pr[y] \cdot \Pr[x | y]$.

$\Pr[M_i(\ )$ ok $| \underbrace{M_i(1^{n_{j'}})\text{ ok}}_{\text{fixed}}$ $\forall j' < j$ $| L_1]$

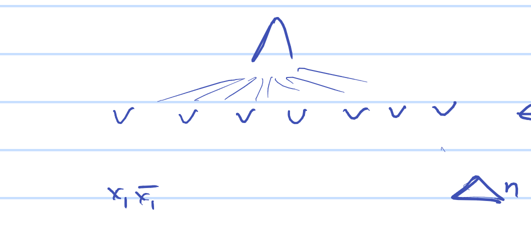
$\cancel{\mathbb{E}_{L_1 \sim D}}$ $\quad \mathbb{E}_{L_1 \sim D'}$

$\Pr[\bigwedge_{i \geq 1} E_i] = \Pr[E_1] \cdot \Pr[\bigwedge_{i \geq 2} E_i | E_1] = \dots$

(3). Show that $\forall f : \{0,1\}^n \to \{0,1\}$. $\exists$ circuit that computes f.
of size

$\to O(2^n \cdot n)$. write f as CNF : $\bigwedge \underbrace{\bigvee y_i}_{\text{or of } n \text{ vars}}$ $\quad y_i = x_i$ / $\bar{x}_i$

$\downarrow$
2^n gates.



$\leftarrow 1$. $\rightsquigarrow 2^n$

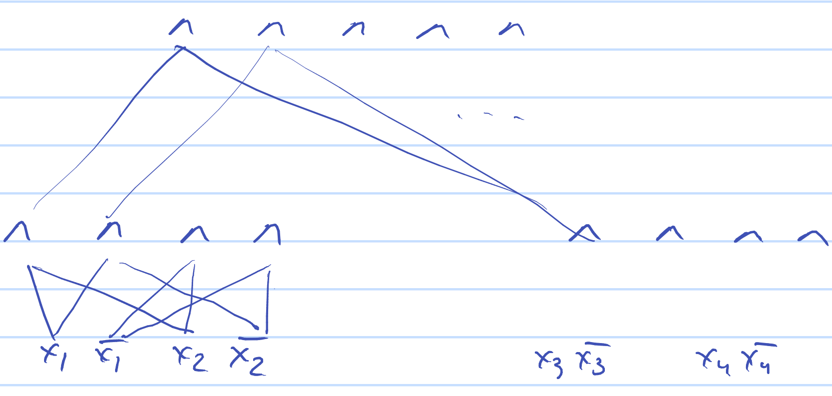
$\leftarrow 2^n$. $\rightsquigarrow 2^n \cdot n$.

Δn

$\to O(2^n)$

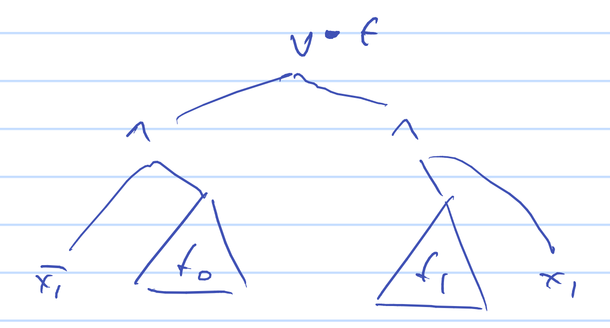
2^{2^n} $\quad \{0,1\}^n$

n vars



$n/2 \cdot 4 + n/4 \cdot 2^4 + n/8 \cdot 2^8 + \dots \underbrace{n/(n/2)}_{2} \cdot 2^{n/2} + n/n \cdot 2^n = O(2^n)$

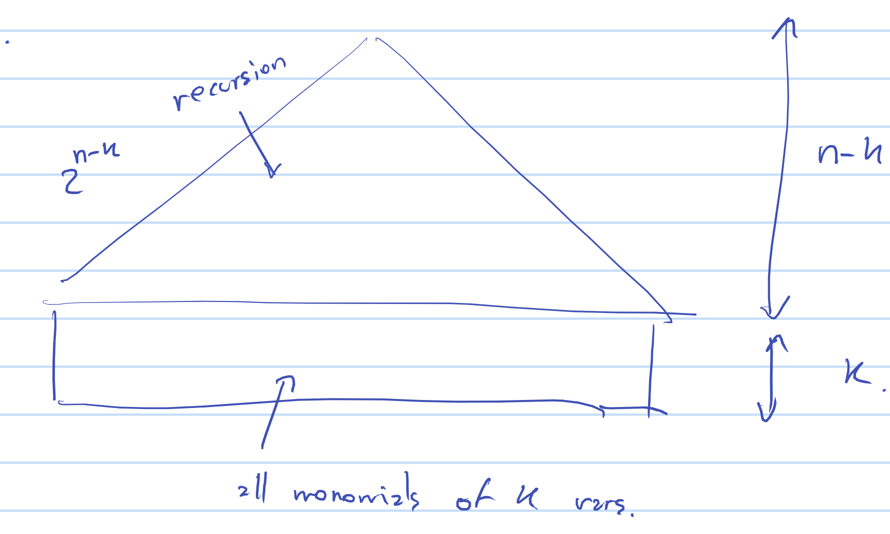
$f(x_1, \dots x_n) = f(0, x_2 \dots x_n) \wedge \bar{x}_1 \vee f(1, x_2 \dots x_n) \wedge x_1$



$T(n) \leq 2 \cdot T(n-1) + 3$

$T(n) = O(2^n)$.

$\to \tilde{O}(2^n/n)$.



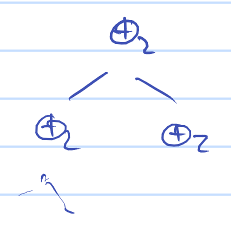
$2^{n-k} + 2^{2^k}$ $\quad$ pick $k = \log(n - \log n)$.

$\downarrow$ $2^n / (n - \log n)$

$\searrow$ $2^{n - \log n} = 2^n/n$.

Parity : $\{0,1\}^n \to \{0,1\}$
$x \mapsto \sum x_i$ mod 2.

$\to$ Compute Parity with a circuit of depth $O(\log n)$, polynomial size.



$\oplus$ constant $\to$ can replace by constant many $\wedge, \vee$ gates.

$\to$ Cannot compute Parity with a circuit of depth 2 of poly size.

assume CNF: $\quad \wedge$
$\vee$ $\vee$ $\vee$ $\vee$.

$\to$ each clause size n.

$\to$ need all clauses. where parity is false. $\to$ negation of each variable.

assume C does not contain $x_1 / \bar{x}_1$. flipping x_1 changes value of parity but not value of circuit.

assume $x_1 = a_1 \dots x_n = b_n$ not present.

assign $x_1 = \bar{a}_1 \dots x_n = \bar{b}_n$. then CNF false but parity true.