1. Show that $NP \neq SPACE(n)$. $\qquad DSPACE(n) = NDSPACE(n)$.

Assume that $NP = SPACE(n)$. we'll show that $NP = SPACE(n^2)$.

this contradicts space hierarchy theorem.

Pick language $L \in SPACE(n^2)$. Pad with 1s.

$L = \{x\}$ $\exists$ TM that given x decides if $x \in L$ in space $O(n^2)$.

$L' = \{(x, 1^m) : x \in L,\ m = |x|^2\}$.

N that given $(x, 1^m)$ simulates M on input x : space $|x^2|$.

size of input is also $|x^2|$

N uses space $O(|\text{input}|)$

$L' \in SPACE(n)$. by hyp $L' \in NP$.

$\exists M$ st $(x, 1^m) \in L' \Leftrightarrow \exists y\ M(x, 1^m, y)$ accepts.

$\exists N$ st given (x, y) simulates $M(x, 1^m, y)$

N accepts iff $x \in L$ $\quad L \in NP$.

by hyp $L \in SPACE(n)$.

2. $UniqueSAT = \{\varphi : \exists! \alpha \text{ st } \varphi(\alpha) = 1\}$. Show that $UniqueSAT \in P^{SAT}$.

call oracle with φ on tape, get 1.

consider $\varphi' = \varphi \wedge \{x_1 = 1\}$, call oracle with φ'. if answer is 1 then set $b_1 = 1$; if answer is 0 then set $b_1 = 0$.

$\varphi^{(n)} = \varphi \wedge \{x_1 = b_1, x_2 = b_2 \dots x_n = 1\}$.

now $\alpha = (b_1 \dots b_n)$ is a satisf. assgn. for φ.

(1) $\left\{ \begin{array}{l} \varphi' = \varphi \wedge \{x_1 \neq b_1\}. \\ \varphi^{(n)} = \varphi \wedge \{x_n \neq b_n\} \end{array} \right.$ | if answer is 1 for any of these calls then $\exists$ another assignment $\beta = \alpha$. so $\varphi \notin UniqueSAT$ o/w $\varphi \in$ —.

or

(2) $\left\{ \begin{array}{l} \xi = \varphi \wedge \text{not}(x_1 = b_1 \wedge x_2 = b_2 \wedge \dots x_n = b_n) \equiv \\ \equiv \varphi \wedge (x_1 \neq b_1 \vee x_2 \neq b_2 \vee \dots x_n \neq b_n). \end{array} \right.$ //

3. Have TM $M^{A,B}$ either A or B is an oracle for TQBF.

but do not know which. Solve TQBF with $M^{A,B}$. $\qquad$ 3CNF 3SAT

Have input $\exists x_1 \forall x_2 \dots Q^n x_n\ \varphi(x_1 \dots x_n).] \ y$

call $A(y)$, $B(y)$. if agree $\to$ ✓. if disagree:

$A = 0, B = 1$. $\qquad \forall x_2 \dots Q^n x_n\ \varphi(0, x_2 \dots x_n)] \ y^0$

$\forall x_2 \dots Q^n x_n\ \varphi(1, x_2 \dots x_n)] \ y^1$

call A, B on y^0, y^1 must have: $A(y^0) = 0$, $A(y^1) = 0$

(at least) one of $B(y^0)$, $B(y^1) = 1$

o/w already caught the liar.

smaller formula (on n-1 vars) where A and B answer differently.

Keep repeating until have formula with no variables.

clearly T or F. then one machine is lying and we know which.

$\boxed{\begin{array}{l} \exists \ \ 2\ 0s,\ \geq 1\ 1 \\ \forall \ \ 2\ 1s,\ \geq 1\ 0. \end{array}}$

4. $\mathcal{D}$: for each n, with prob. 1/2 no string of length n is in L.
with prob 1/2 a uniform string of length n is in L.
exactly one

Show that $\Pr_{L \sim \mathcal{D}}[P^L \neq NP^L] \to 1$.

$L \to UL \qquad UL = \{1^n : \exists x \in L, |x| = n\}$.

whp no machine in P^L can decide UL.

$\Pr[\exists i\ M_i^L \text{ decides } L] \leq \sum \underbrace{\Pr[M_i^L \text{ decides } L]}_{0}$

whp M_i^L make an error. $\qquad$ (full solution next week).

5. ExactClique is lang. $\{(G, k)$ st largest clique in G has size exactly $k\}$.

Show that Exact Clique is in $\Pi_2 P$

Exact Clique is $\{(G, k)$ st $\forall S\ |S| = k+1\ \exists T\ |T| = k$
S not a clique and T is a clique$\}$.

6. DP is set of languages $L = L_1 \cap L_2$ with $L_1 \in NP$, $L_2 \in coNP$
$\to L = L_1' \cap \overline{L_2'}$ with $L_1', L_2' \in NP$

Show that Exact-Clique is DP-complete.

Exact Clique $\in$ DP. $\quad L_1\ \{(G, k) \quad G \text{ has } k\text{-clique}\}$
$L_2\ \{(G, k) \quad G \text{ has no } k+1\text{-clique}\}$.

pick $L = L_1 \cap \overline{L_2}$ $\quad L_1 \in NP$, $L_2 \in NP$. $\quad$ clique is NPc

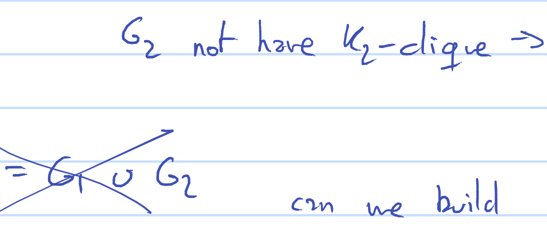
$f : L_1 \to Clique$ $\qquad g : L_2 \to Clique$
$x \mapsto (G_1, k_1)$ $\qquad x \mapsto (G_2, k_2)$. $\qquad$ build input for exact k-clique.

$k_1 = k_2$?

$k_1 \to k$

$k_2 \to k$.

k_2^2 edges



k_2

$\boxed{\begin{array}{l} G_1 \text{ has } k \text{ clique} \\ G_2 \text{ does not have } k{+}1\text{-clique.} \end{array}}$

replace every vertex with k_2-clique.

claim: G_1 had k_1-clique $\to G_1'$ have $k_1 \cdot k_2$ - clique.

G_2 not have k_2-clique $\to G_2'$ does not have $k_1 \cdot k_2$-clique.

$\cancel{G = G_1 \cup G_2}$ $\qquad$ can we build G st G has largest clique of sz exactly k iff G_1 has k clique, G_2 does not have $k{+}1$-clique.

Tensor product: $G = ((u, v)$ with $u \in G_1'$ $v \in G_2')$.

edge $((u_1, v_1), (u_2, v_2))$ iff
edge $(u_1, u_2) \wedge$ edge (v, v_2).

$clique(G) = \min(G_1, G_2)$.

$S \subseteq V(G_1) \qquad (s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$
$T \subseteq V(G_2)$

$G_1 \geq k$.

G_2 no clique of sz $k+1$ $(\leq k)$.

$G_2'' = G_2' \cup K_k$. $\qquad G_2''$ has clique of sz exactly k.

G_1' has clique of sz $\geq k$.

$G_1' \times G_2''$ has clique of sz exactly k.

problem: what if G_2'' has large clique
and G_1' has clique of sz k $\Big]$? $\quad$ needs fixing!