Hard Examples for Common Variable Decision Heuristics

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Banff workshop on Proof Complexity

DPLL

Algorithm 1: DPLL
while *not solved* **do**
    if *conflict* **then** backtrack()
    else if *unit* **then** propagate()
    else
       decide()

State: partial assignment

CDCL

Algorithm 2: CDCL
while *not solved* **do**
    if *conflict* **then** learn()
    else if *unit* **then** propagate()
    else
        maybe forget()
        maybe restart()
        decide()

State: partial assignment & learned clauses

CDCL equivalent to Resolution

Theorem   [Beame, Kautz, Sabharwal '04]

Resolution p-simulates CDCL

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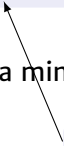Resolution p-simulates CDCL

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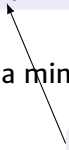**Theorem**   [Pipatsrisawat, Darwiche '09]

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Also: CDCL with random decisions simulates bounded-width Resolution
                                                    [Atserias, Fichte, Thurley '09].

Separation of CDCL vs Resolution

Theorem

There are formulas such that

- ▶ Resolution refutations of polynomial length
- ▶ Exponential time in CDCL with common decision heuristics

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Which literal do we pick next?

▶ Will lead to a conflict quickly.
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▶ Give a score $q(x)$ to variable x.
▶ At each conflict
  ▶ Bump $q' = q + 1$ if x involved.
  ▶ Decay $q' = 0.95 \cdot q$ all variables.
▶ Pick variable with largest score

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▶ Last assigned.

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Properties of VSIDS

▶ Each conflict
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Fine Print
Not true if finite precision.
Does hold if stable priority queue.

Separation of CDCL vs Resolution

Definition

A decision heuristic rewards conflicts if
a variable involved in a conflict is picked before a variable that never has.

Theorem

There are formulas such that
- ▶ Resolution refutations of polynomial length
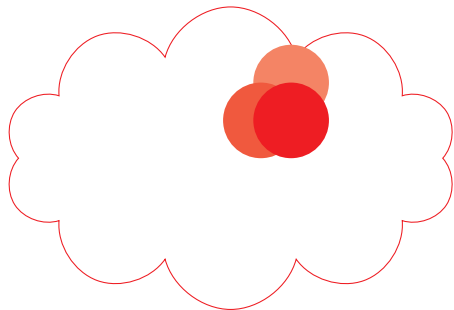- ▶ Exponential time in CDCL with conflict-rewarding heuristics

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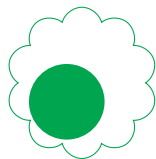
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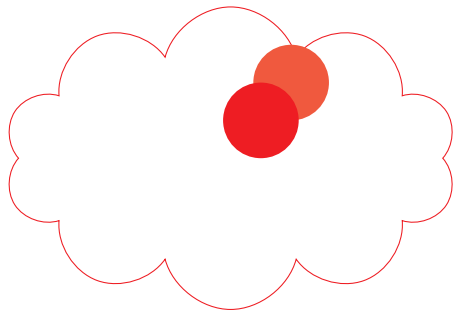
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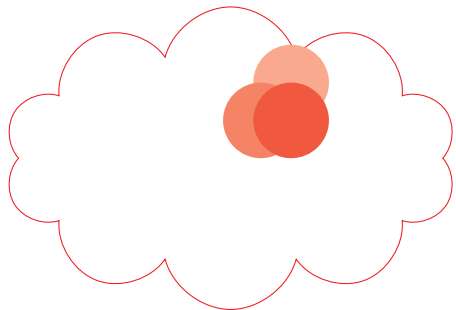
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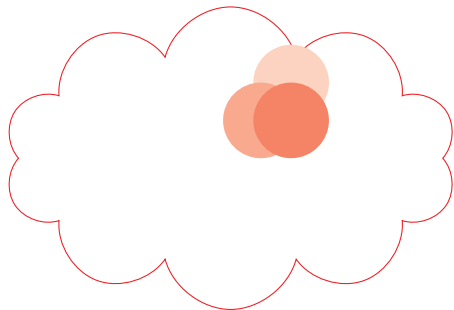- Eventually stabilize.
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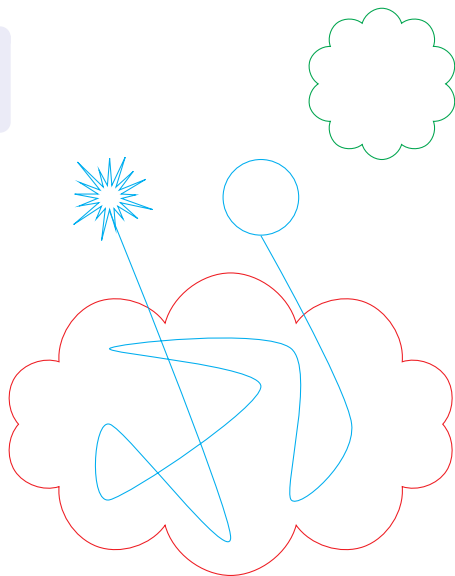- Decay $q' = 0.95 \cdot q$ all variables.

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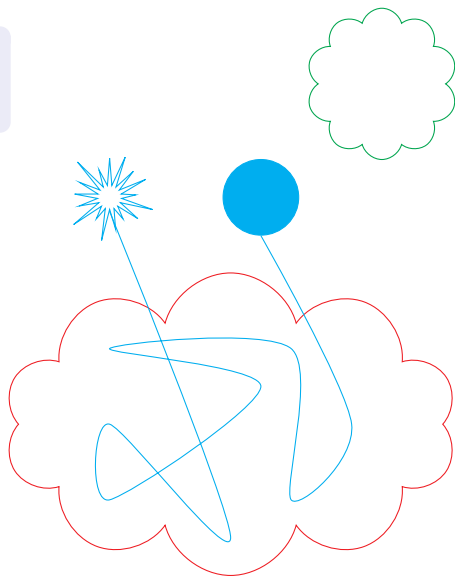- Eventually stabilize.
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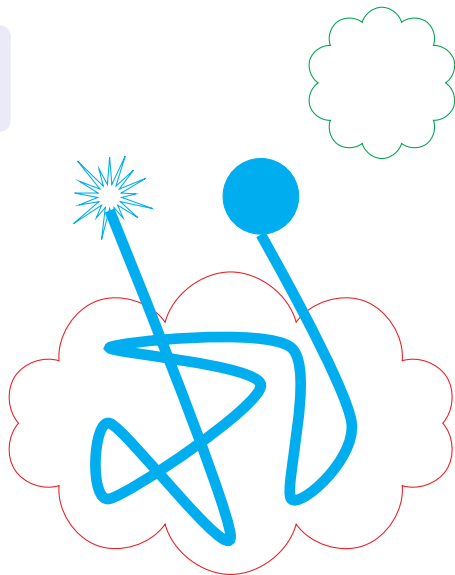▶ Pitfall gadget produces a conflict involving all hard variables.

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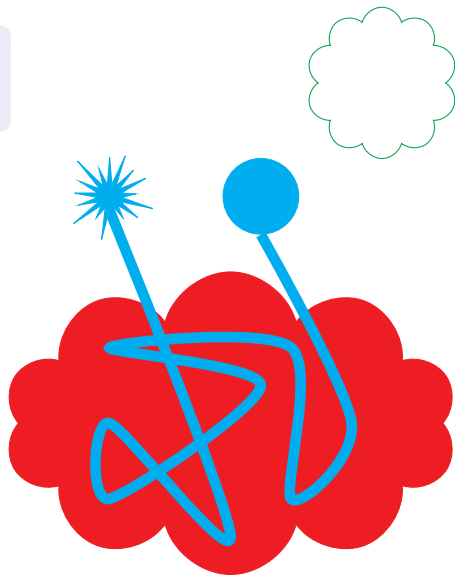▶ Pitfall gadget produces a conflict involving all hard variables.

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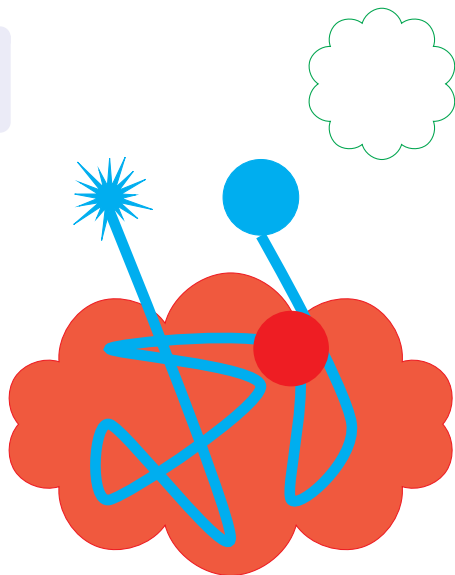- Pitfall gadget produces a conflict involving all hard variables.
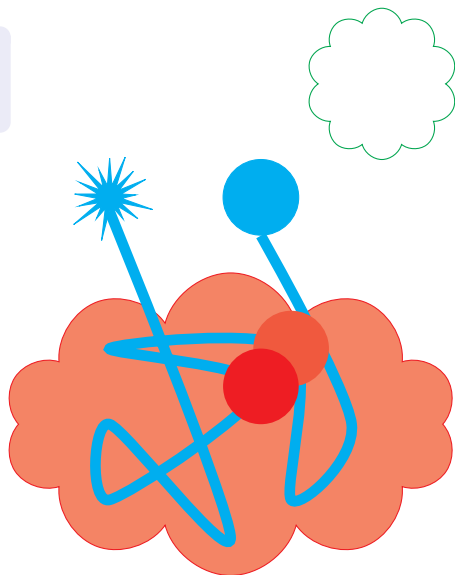- Solver stuck with hard variables!

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- **Pitfall gadget** produces a conflict involving all hard variables.
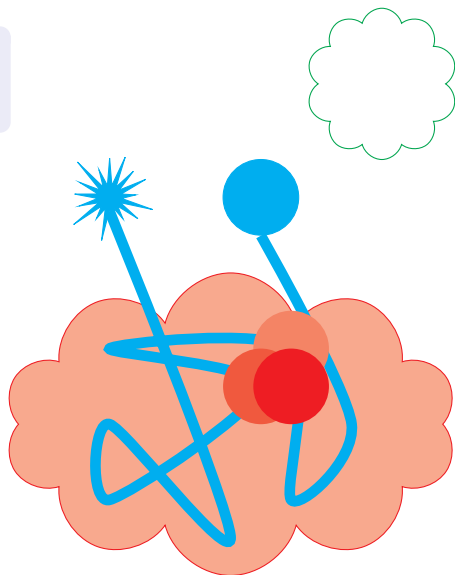- Solver stuck with hard variables!

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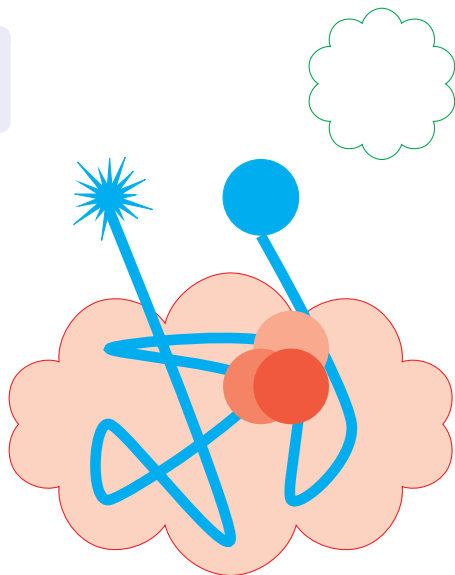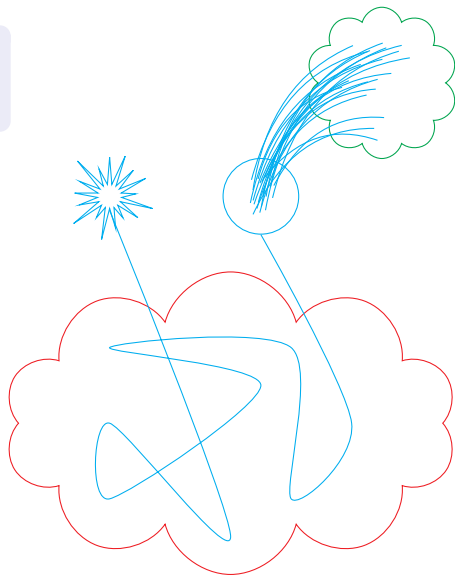
▶ Solver stuck with hard variables!

▶ But still $1/\text{poly}$ probability of solving easy part first.

Intuition (III)

- ▶ Bump $q' = q + 1$ if x involved.
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- ▶ Make easy variables lead to pitfall gadget.

Formula Description

Pitfall Formula Φ

Variables
- X Hard
- Y Easy
- Z Auxiliary

Gadgets
- $\mathrm{Ts}(X, Z)$ Padded Tseitin
- $\Gamma(Y)$ Easy
- $\Psi(Y, Z)$ & $\Pi(Z, X)$ Pitfall
- $\Delta(Z)$ Tail
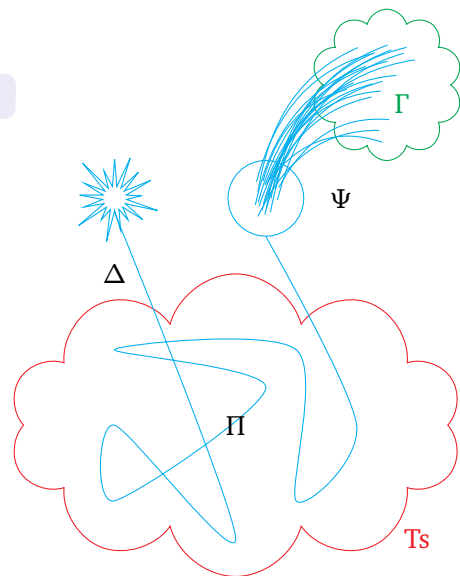
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▶ Hence π exponential.

Proof Sketch (II)

Need to ensure no conflicts use Γ clauses.
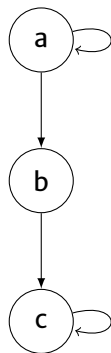Define following solver states:

(a)

- ▶ No conflict
- ▶ No pair of Y variables assigned
- ▶ Enough Z variables unassigned

(b)

- ▶ (a) + a pair of Y variables assigned

(c)

- ▶ (a) + all X variables involved in a conflict

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Thanks!