

Hardness of Approximation in PSPACE and Separation Results for Pebble Games

Marc Vinyals

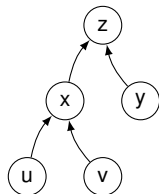
KTH Royal Institute of Technology
Stockholm, Sweden

Joint work with Siu Man Chan,
Massimo Lauria, and Jakob Nordström

56th Annual IEEE Symposium on Foundations of Computer Science

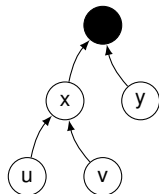
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- ▶ Single player game on DAG
 - ▶ Indegree 2, single sink



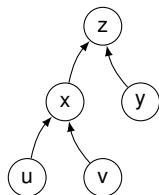
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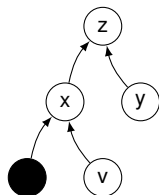
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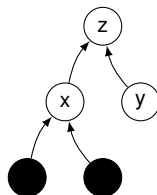
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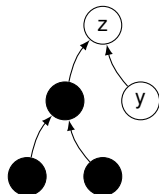
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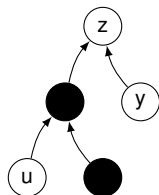
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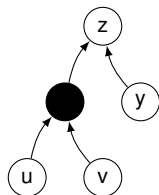
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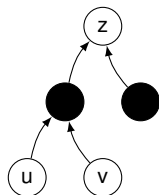
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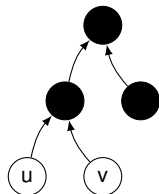
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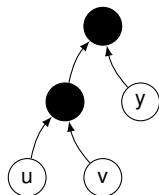
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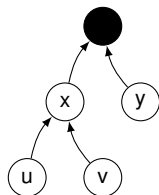
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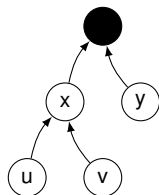
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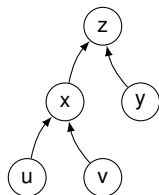
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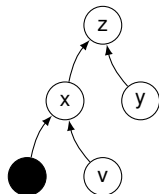
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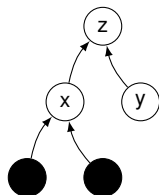
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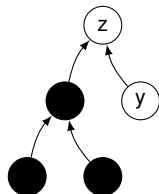
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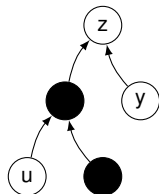
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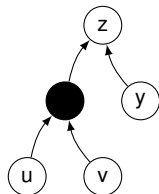
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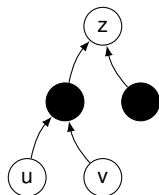
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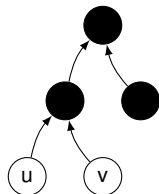
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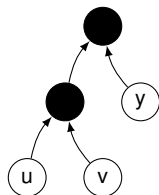
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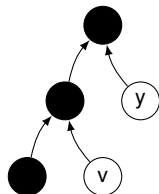
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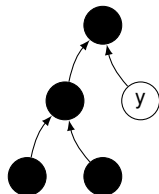
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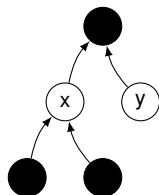
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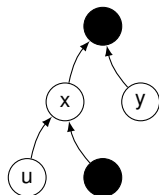
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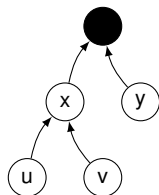
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Connections to other games

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Reversible pebbling equivalent to Dymond–Tompa and Raz–McKenzie.
[Chan ’13]

Why Study Pebbling?

- ▶ Flowcharts and recursive schemata [Paterson, Hewitt '70]
- ▶ Register allocation [Sethi '75]
- ▶ Time and space trade-offs [Chandra '73; Cook '74; Hopcroft, Paul, Valiant '77; Savage, Swamy '77,'79,'83; Tompa '78]
- ▶ Parallel time [Dymond, Tompa '85; Venkateswaran, Tompa '89]
- ▶ Communication complexity [Raz, McKenzie '99]
- ▶ Monotone space complexity [Chan, Potechin '12; Filmus, Pitassi, Robere, Cook '13]
- ▶ Proof complexity [Bonet, Esteban, Galesi, Johannsen '98; Ben-Sasson, Wigderson '99; Ben-Sasson, Nordström '08, '11]
- ▶ Cryptography [Dwork, Naor, Wee '05; Alwen, Serbinenko '15]

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Interesting combinatorial game in itself

Standard vs Reversible Pebbling

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[Královič '01]

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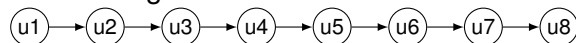
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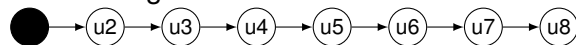
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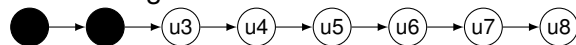
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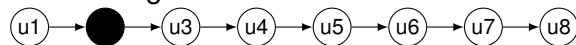
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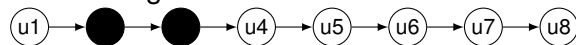
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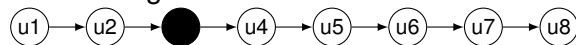
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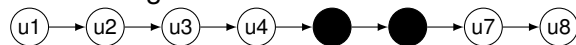
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Standard vs Reversible Pebbling

Simulation

[Královič '01]

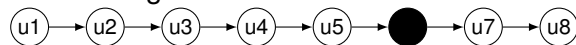
Standard price $p \rightarrow$ reversible price $O(p^2 \log n)$

Separation

[Bennett '89]

Reversible price at least $\max(p, \log \text{depth})$

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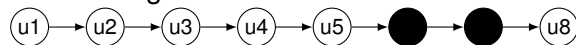
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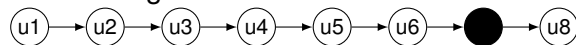
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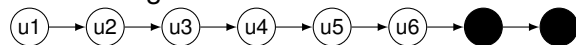
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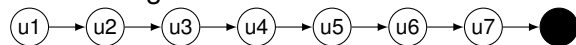
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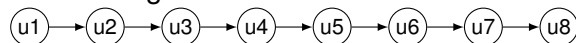
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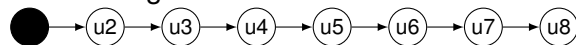
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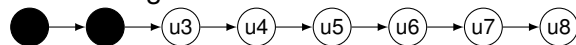
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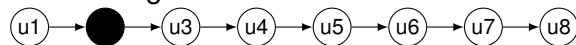
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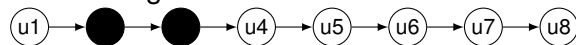
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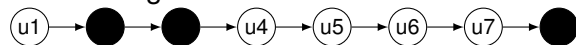
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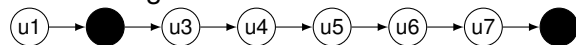
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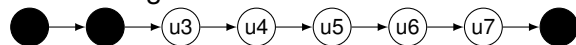
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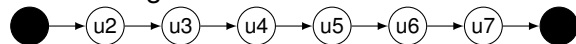
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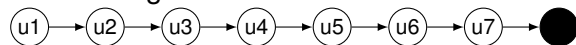
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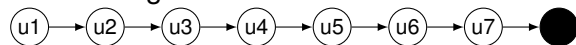
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Question

Are standard and reversible prices within additive $\log n$?

Complexity of Pebbling

Can we pebble a given graph with p pebbles?

Standard pebbling is PSPACE-complete [Gilbert, Lengauer, Tarjan '80]

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Is reversible pebbling PSPACE-complete?

Yes, if we had unbounded indegree [Chan '13]

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Is there a polynomial time algorithm that computes pebbling price within 1 pebble?

Our Results

Theorem (Separation)

Standard and reversible pebbling are separated by a $\log n$ factor.

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Standard and reversible pebbling are PSPACE-hard to approximate within any additive constant.

Stronger Separation

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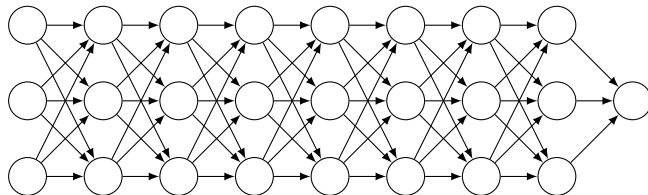
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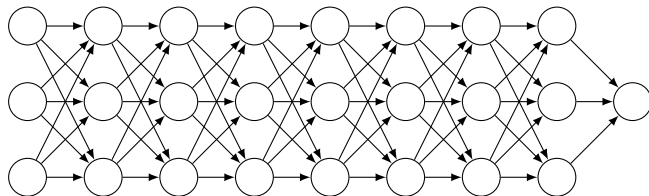


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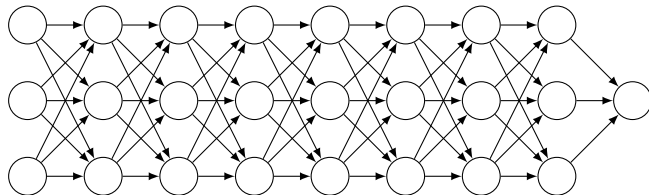
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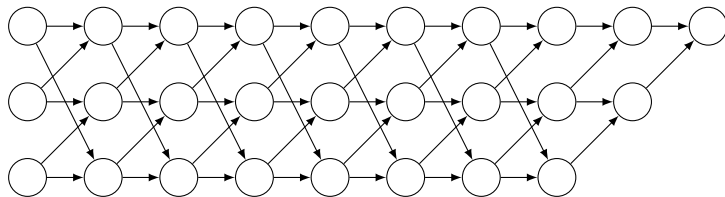
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Road graph

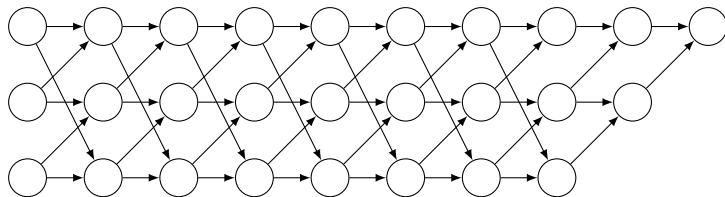


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Road graph



- ▶ Standard price: $p + 2$
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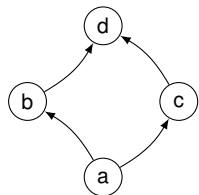
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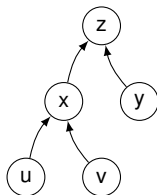
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- ▶ Theorem follows by amplifying gap in decision problem

Product for Reversible Pebbling

Outer Graph O

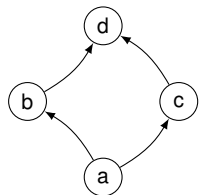


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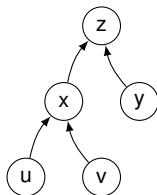


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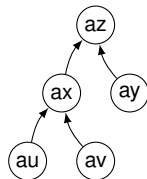
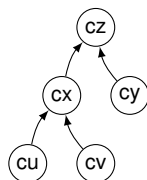
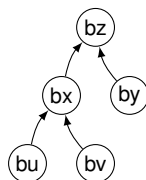
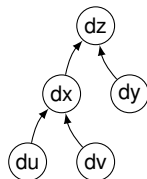
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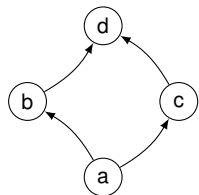


- ▶ One copy of I for each vertex of O

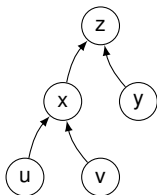


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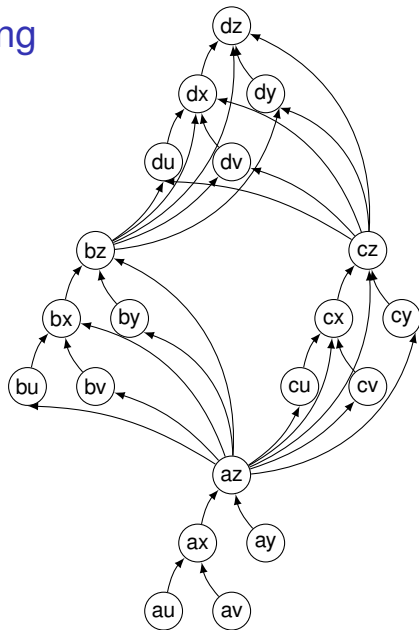
Outer Graph O



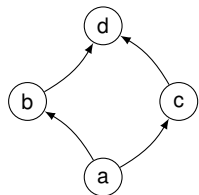
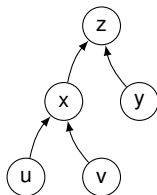
Inner Graph I



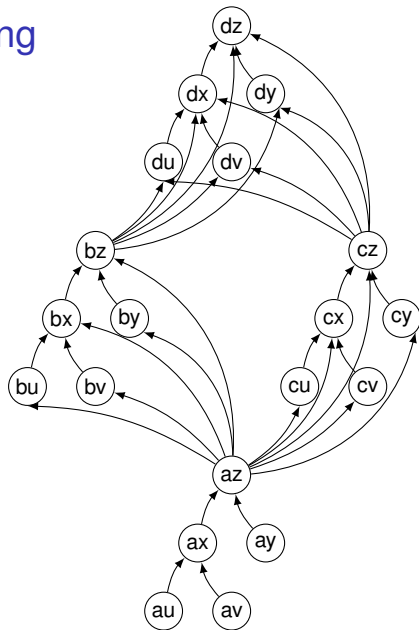
- ▶ One copy of I for each vertex of O
- ▶ $|I|$ edges for each edge of O



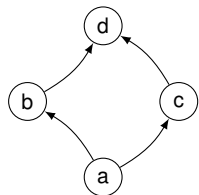
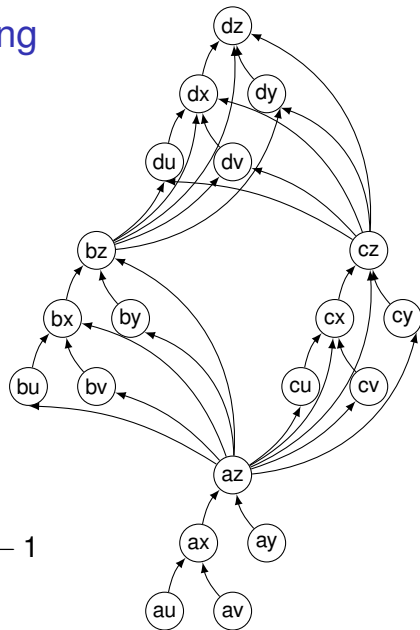
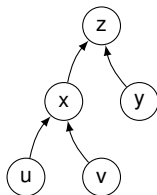
Product for Reversible Pebbling

Outer Graph O Inner Graph I 

- ▶ One copy of I for each vertex of O
- ▶ $|I|$ edges for each edge of O
- ▶ Indegree blows up, will fix later



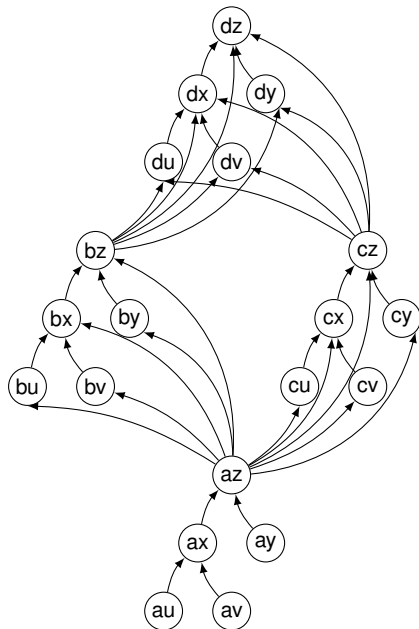
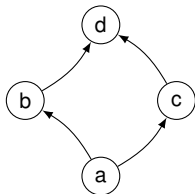
Product for Reversible Pebbling

Outer Graph O Inner Graph I 

- ▶ One copy of I for each vertex of O
- ▶ $|I|$ edges for each edge of O
- ▶ Indegree blows up, will fix later
- ▶ $\text{price}(O \times I) = \text{price}(O) + \text{price}(I) - 1$

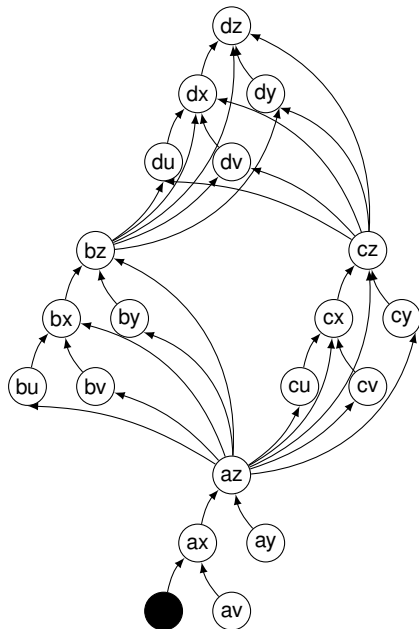
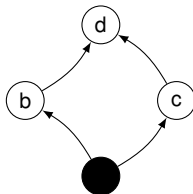
Upper Bound

- ▶ Follow pebbling for outer graph



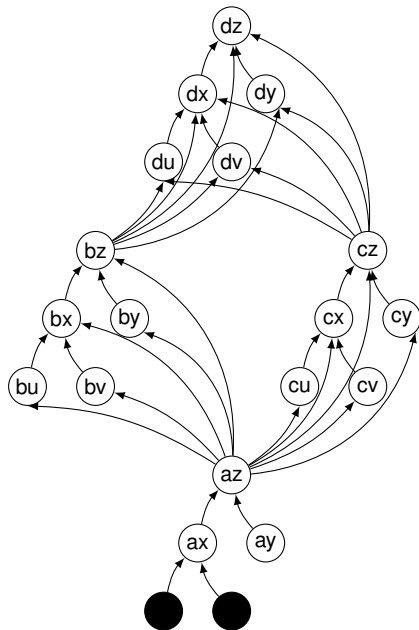
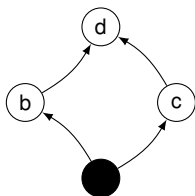
Upper Bound

- ▶ Follow pebbling for outer graph
- ▶ Move \rightarrow pebbling of inner graph



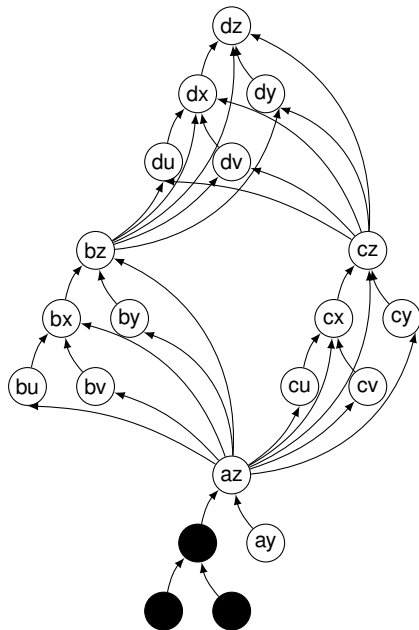
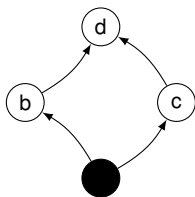
Upper Bound

- ▶ Follow pebbling for outer graph
- ▶ Move \rightarrow pebbling of inner graph



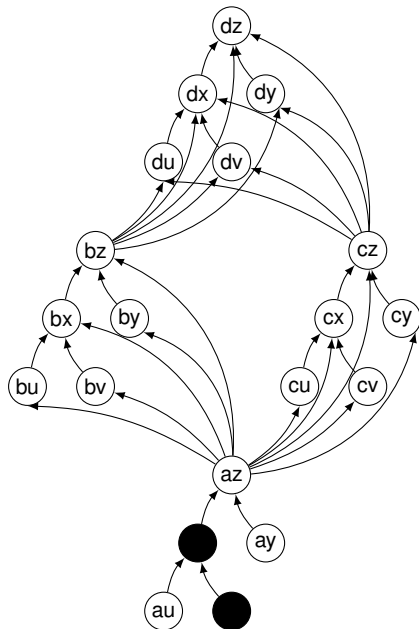
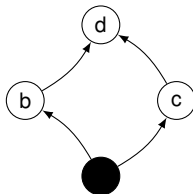
Upper Bound

- ▶ Follow pebbling for outer graph
- ▶ Move \rightarrow pebbling of inner graph



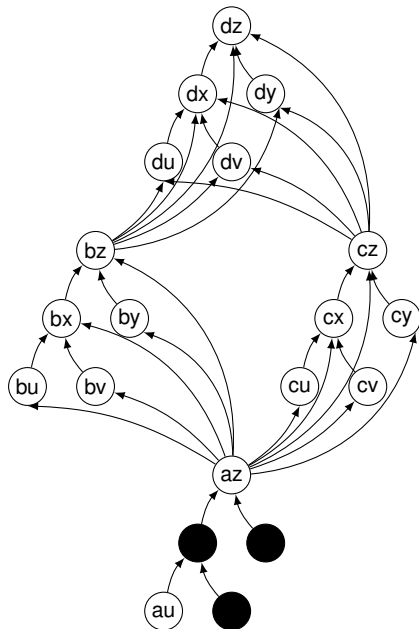
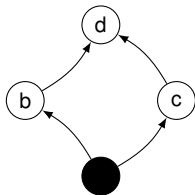
Upper Bound

- ▶ Follow pebbling for outer graph
- ▶ Move \rightarrow pebbling of inner graph



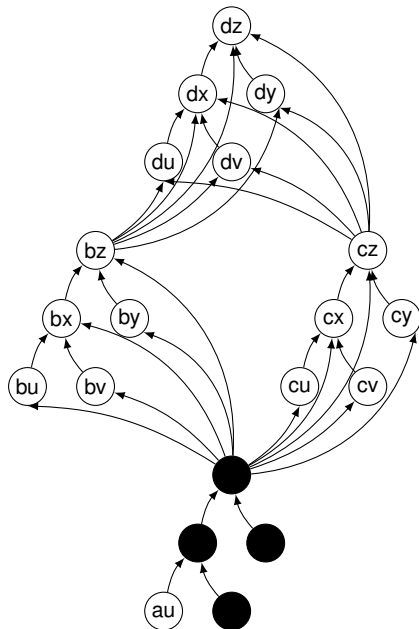
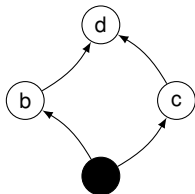
Upper Bound

- ▶ Follow pebbling for outer graph
- ▶ Move \rightarrow pebbling of inner graph



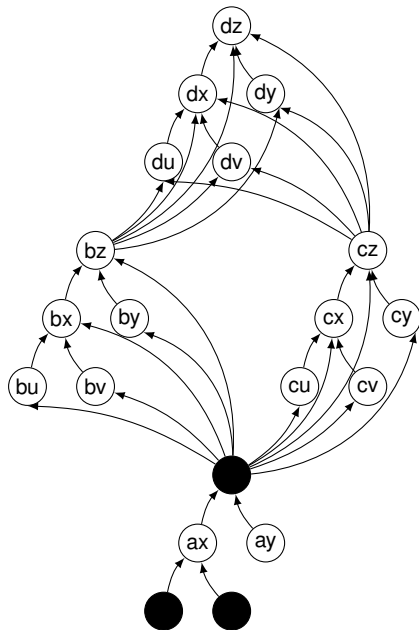
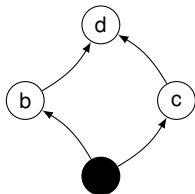
Upper Bound

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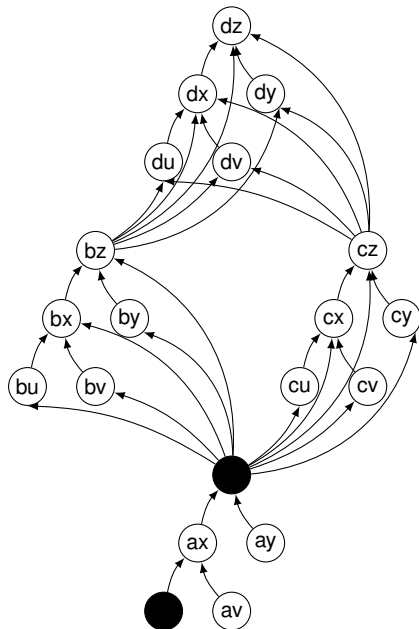
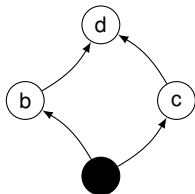
Upper Bound

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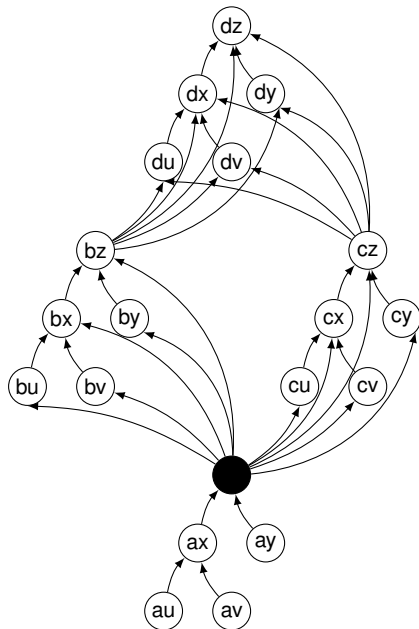
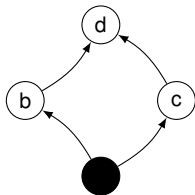
Upper Bound

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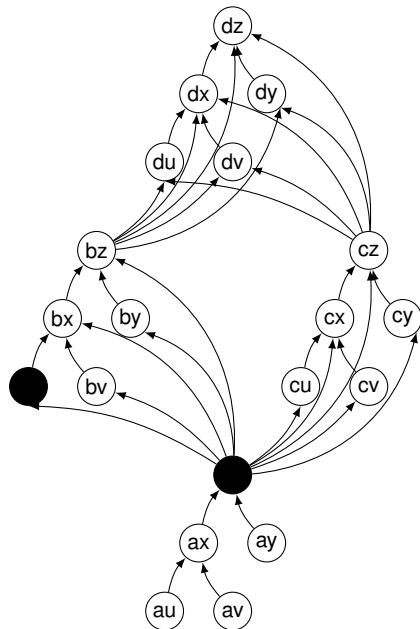
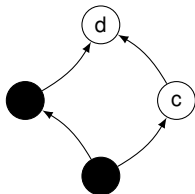
Upper Bound

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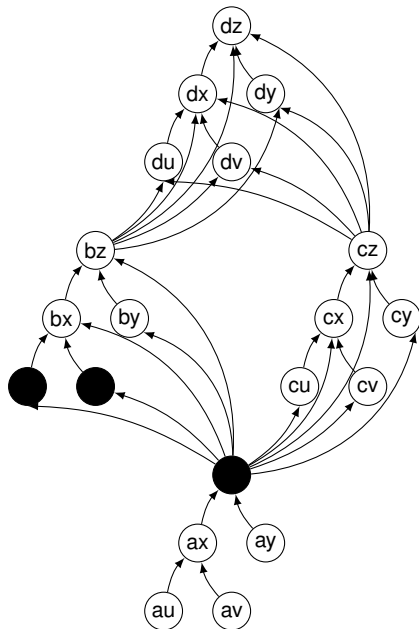
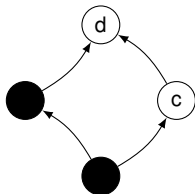
Upper Bound

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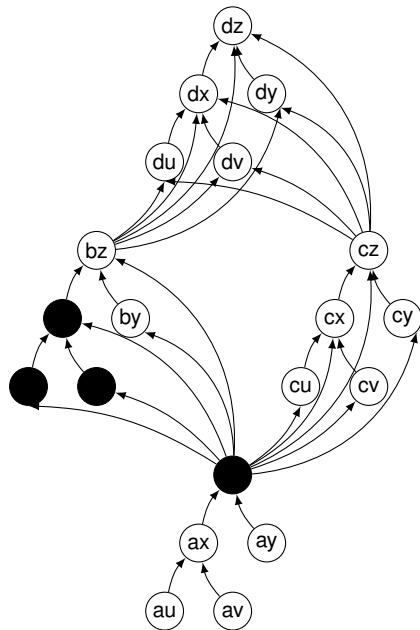
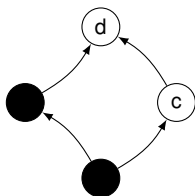
Upper Bound

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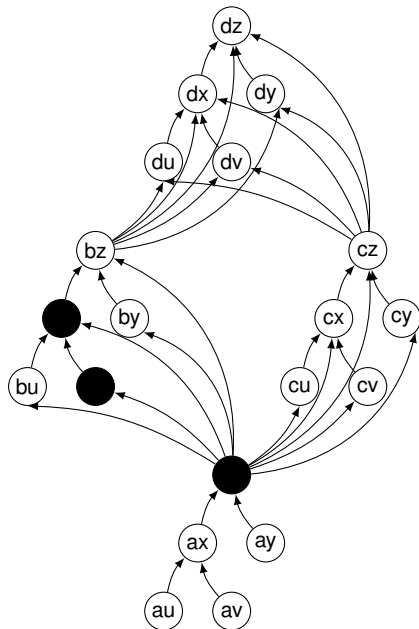
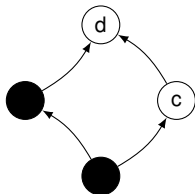
Upper Bound

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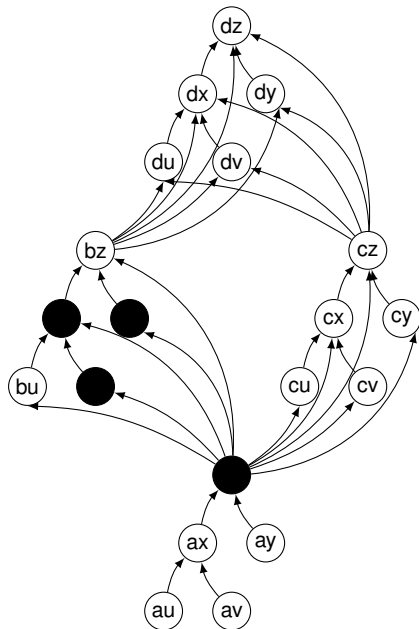
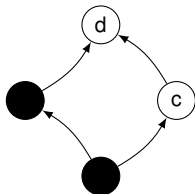
Upper Bound

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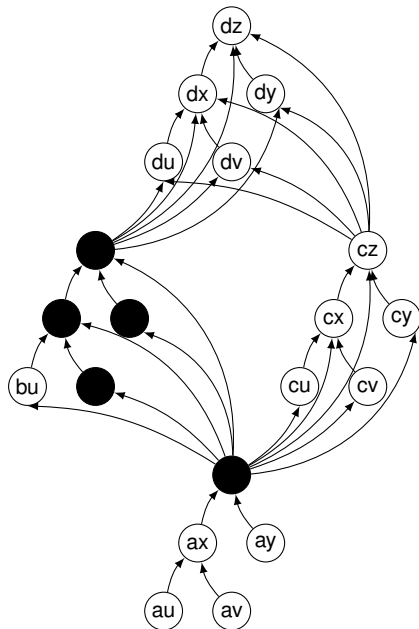
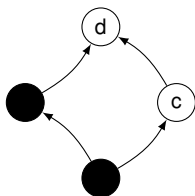
Upper Bound

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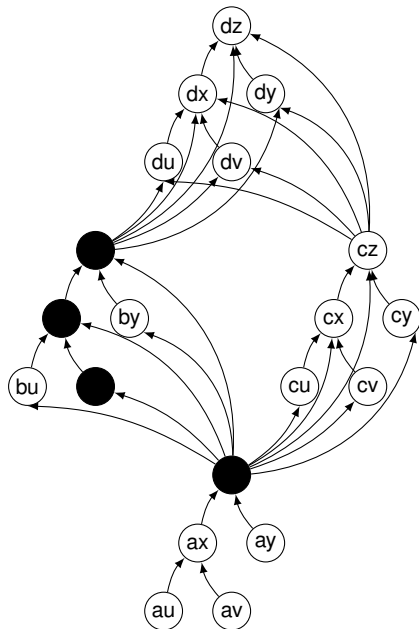
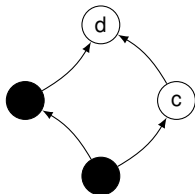
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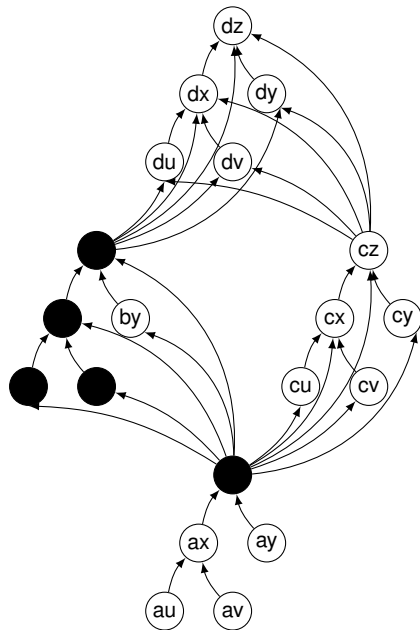
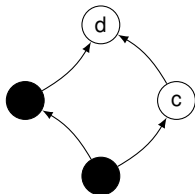
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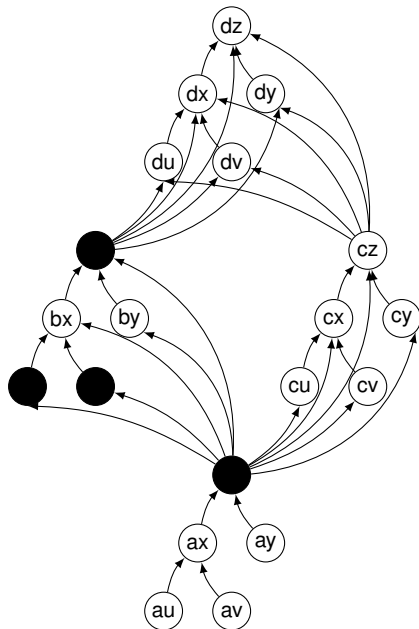
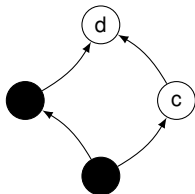
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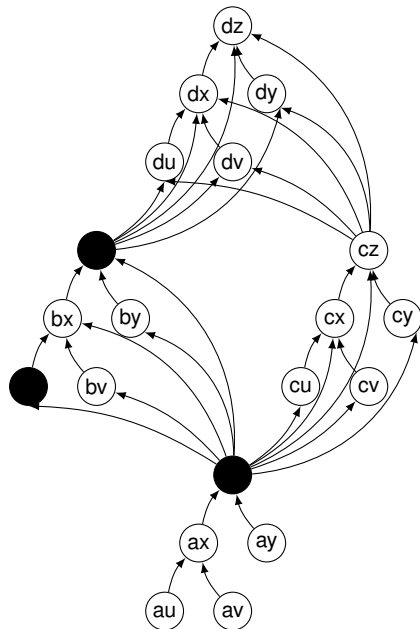
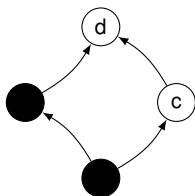
Upper Bound

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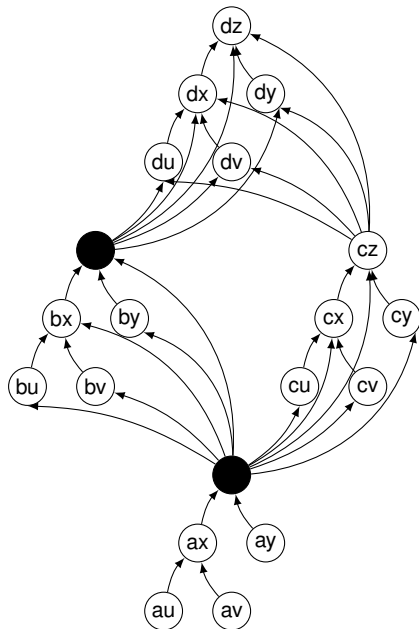
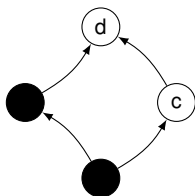
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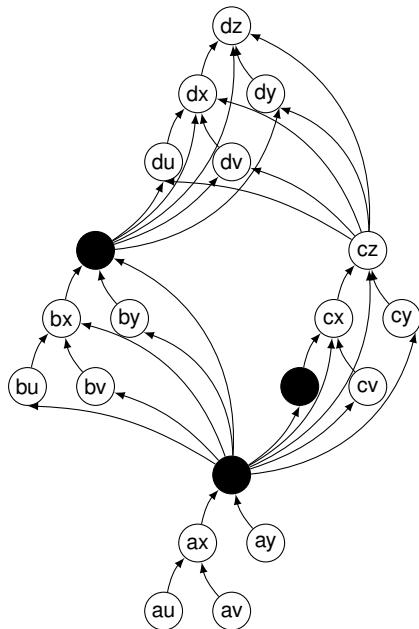
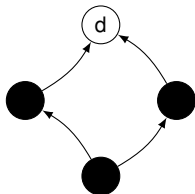
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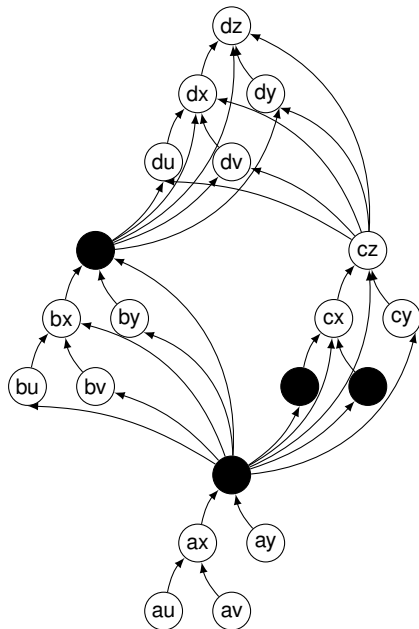
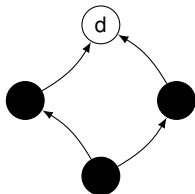
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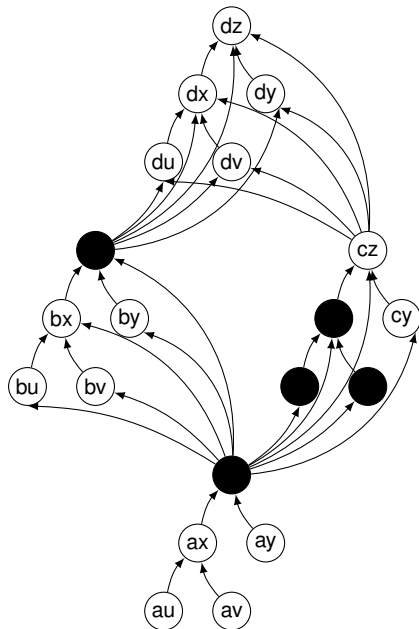
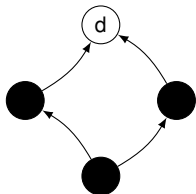
Upper Bound

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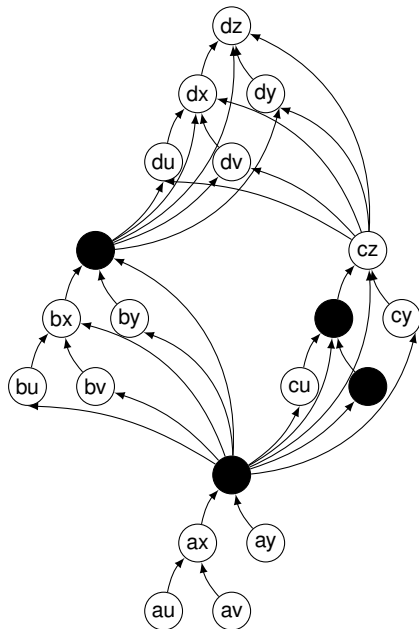
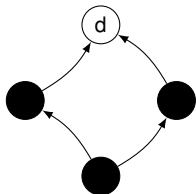
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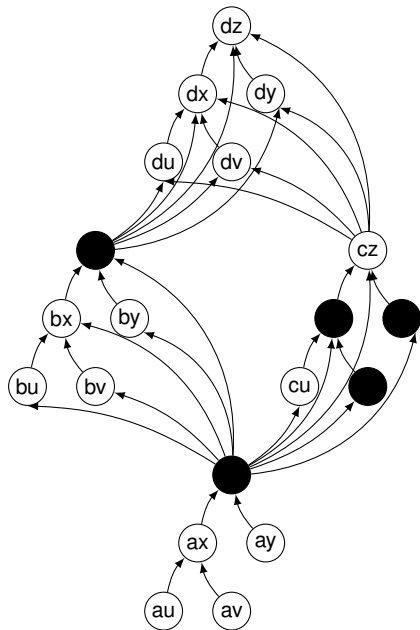
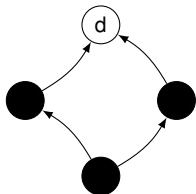
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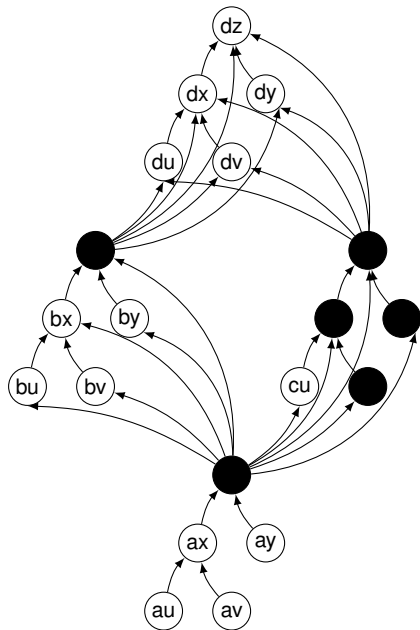
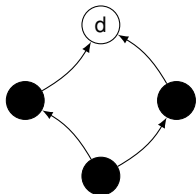
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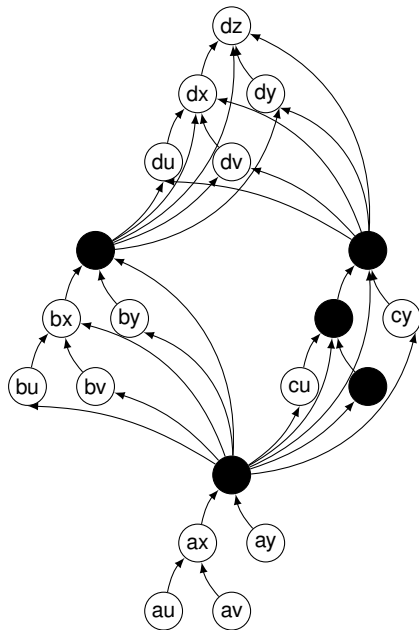
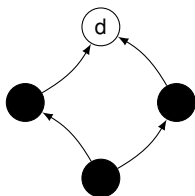
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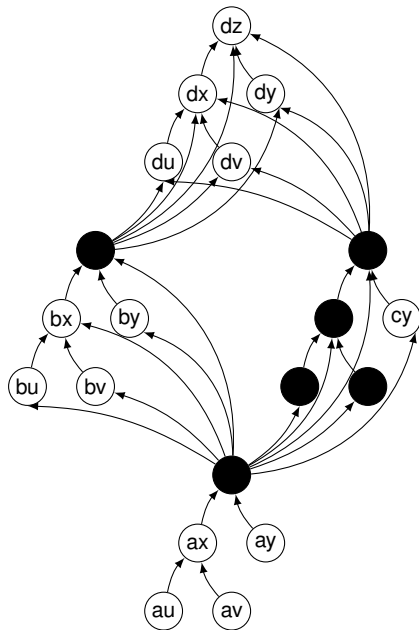
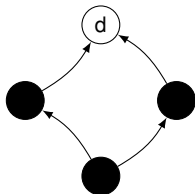
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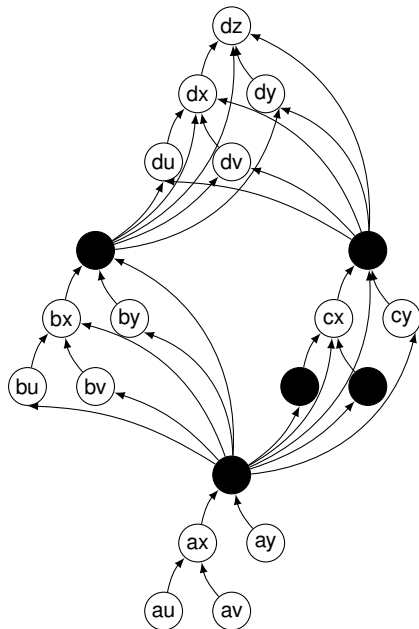
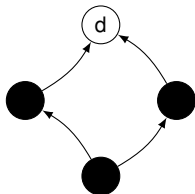
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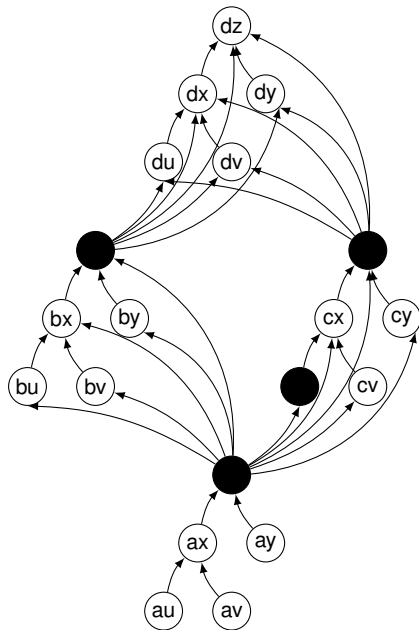
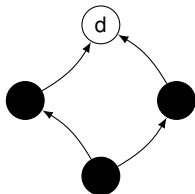
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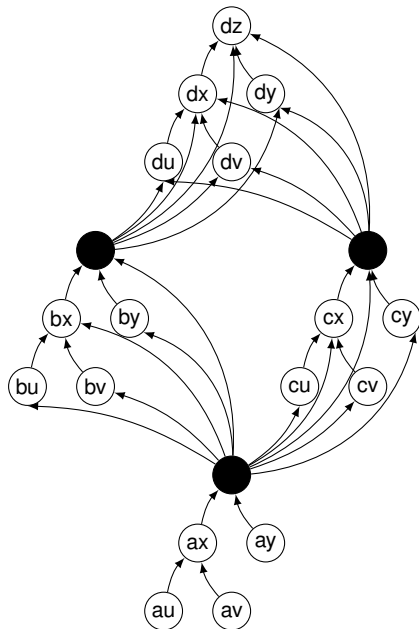
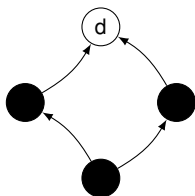
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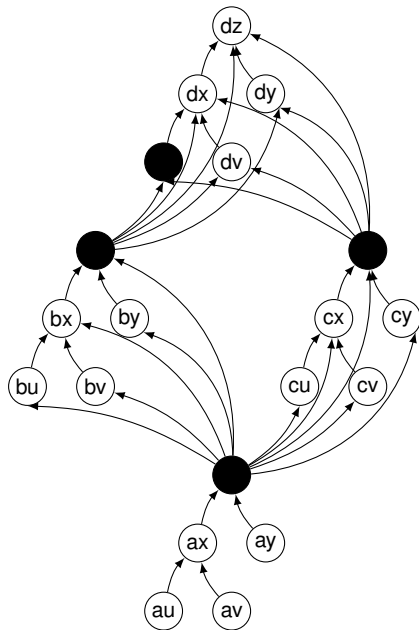
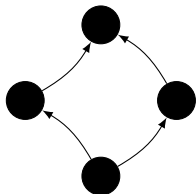
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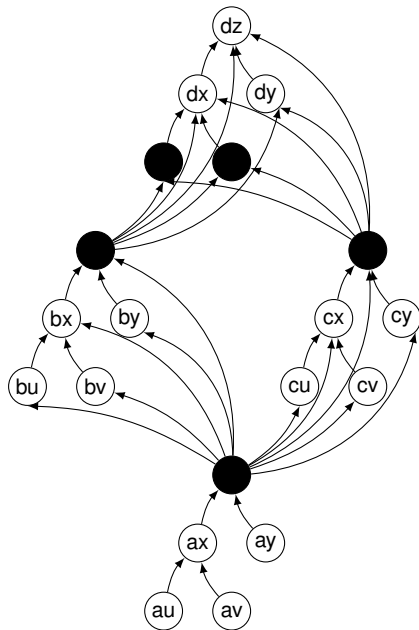
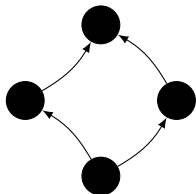
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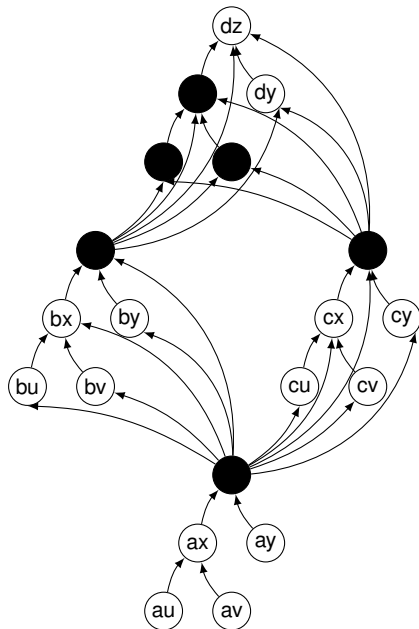
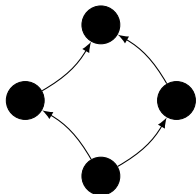
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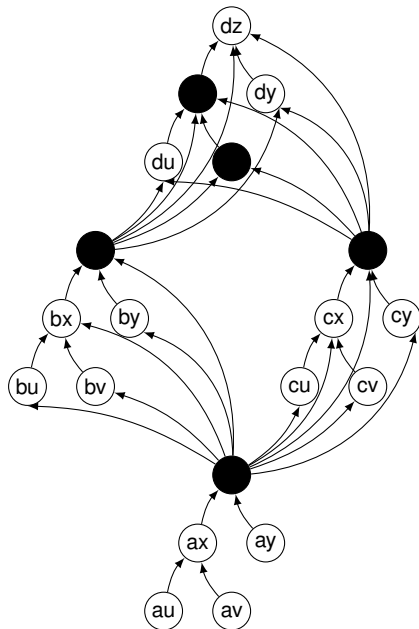
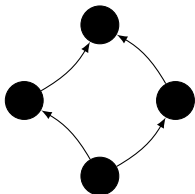
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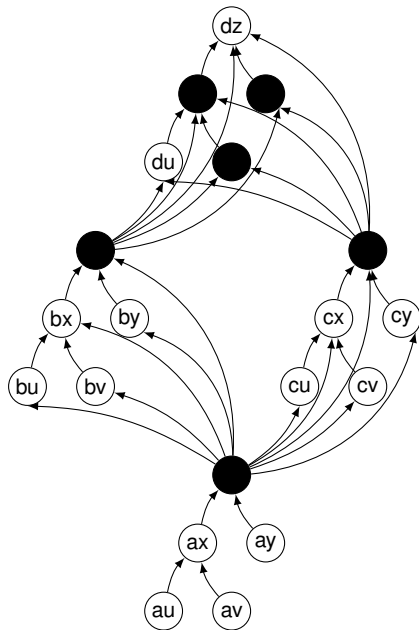
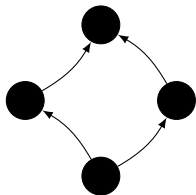
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- ▶ Move \rightarrow pebbling of inner graph



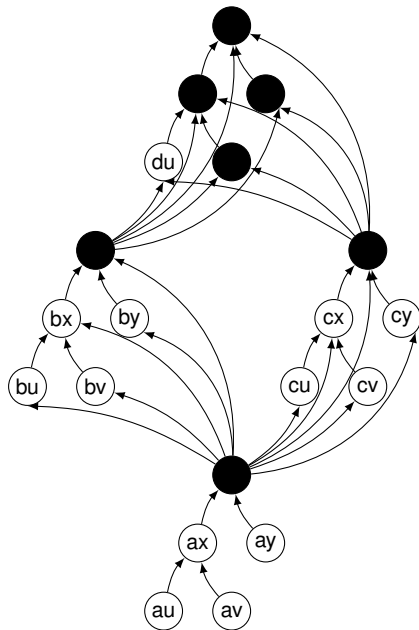
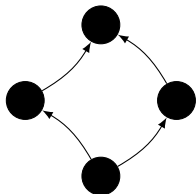
Upper Bound

- ▶ Follow pebbling for outer graph
- ▶ Move \rightarrow pebbling of inner graph



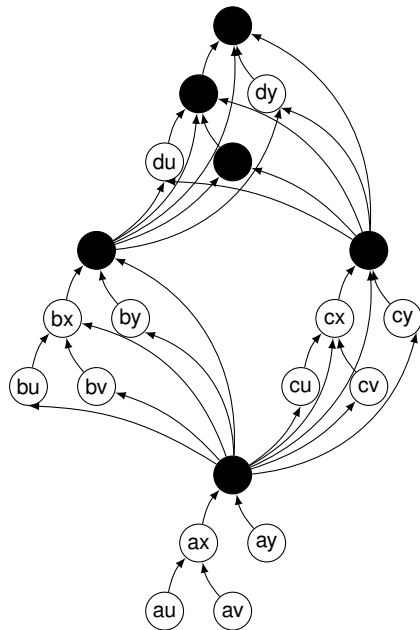
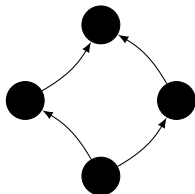
Upper Bound

- ▶ Follow pebbling for outer graph
- ▶ Move \rightarrow pebbling of inner graph
- ▶ $\text{price}(O) + \text{price}(I) - 1$ pebbles



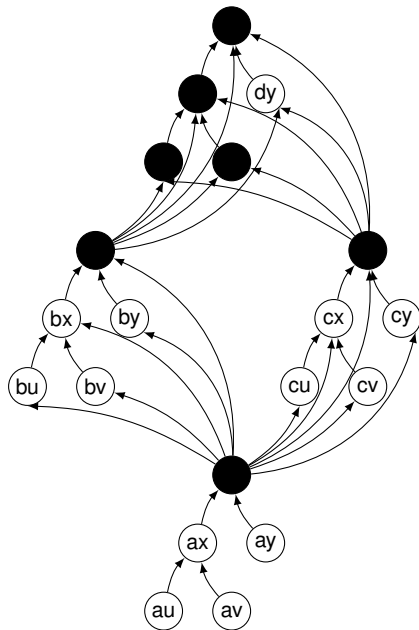
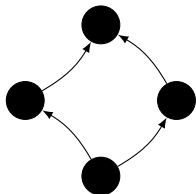
Upper Bound

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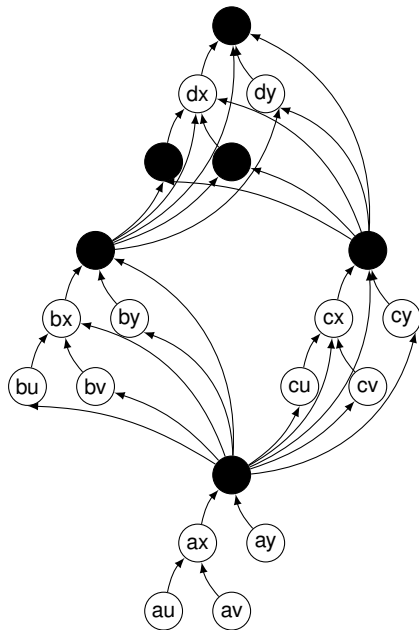
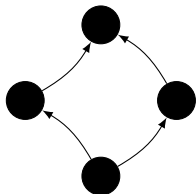
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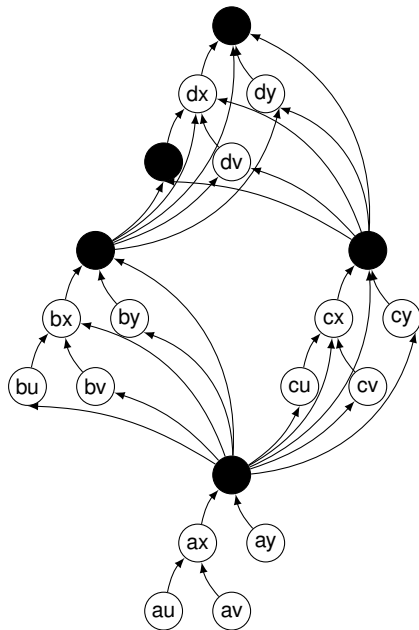
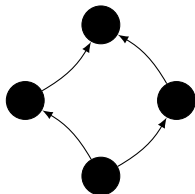
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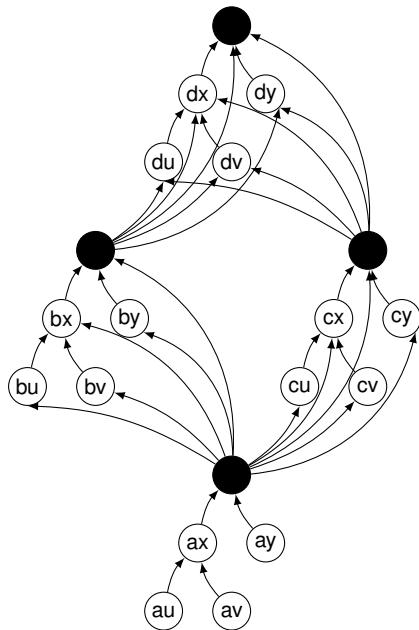
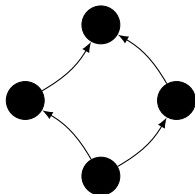
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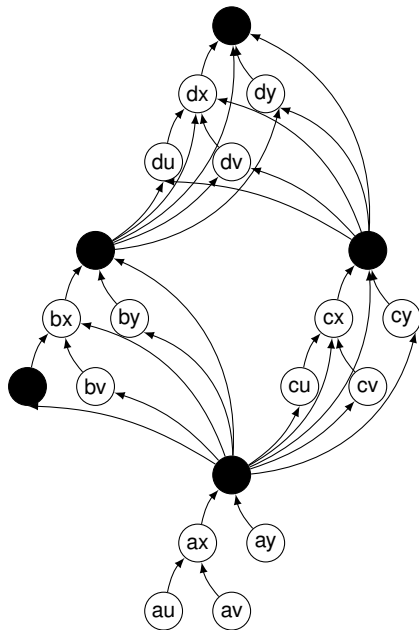
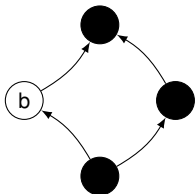
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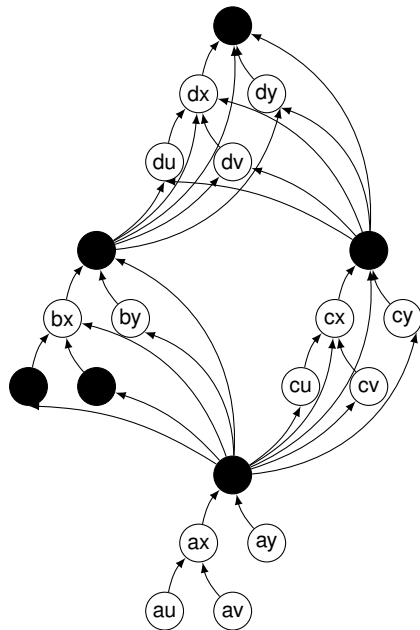
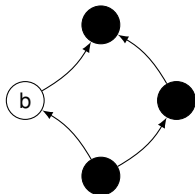
Upper Bound

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- ▶ $\text{price}(O) + \text{price}(I) - 1$ pebbles



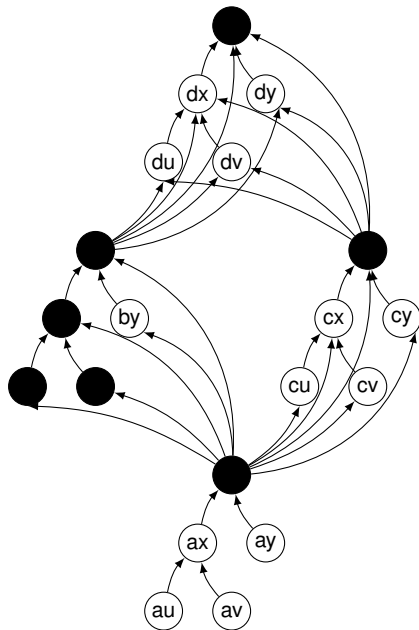
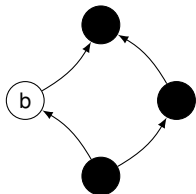
Upper Bound

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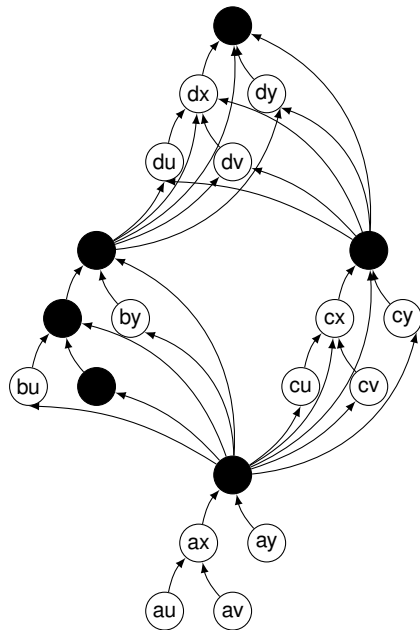
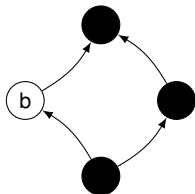
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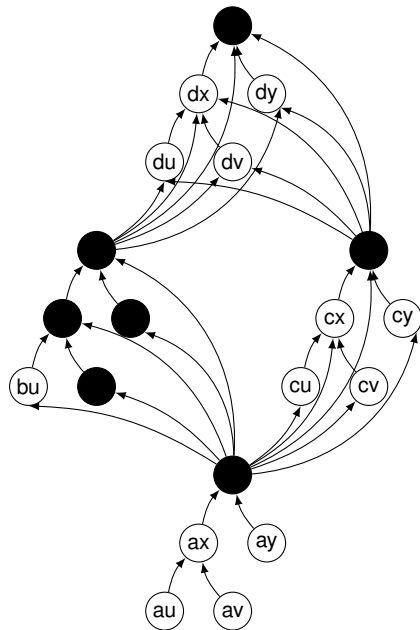
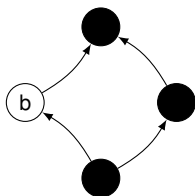
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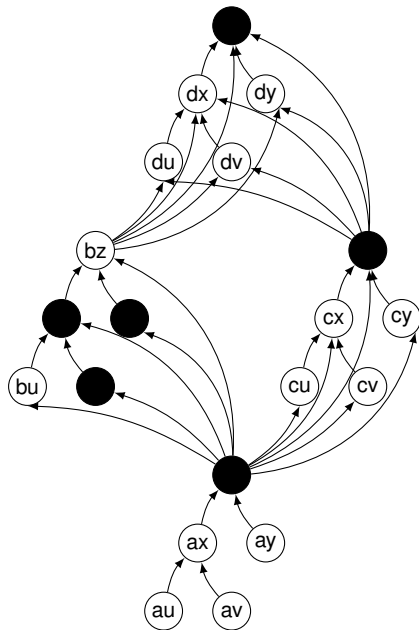
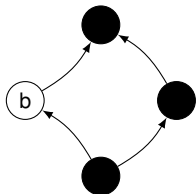
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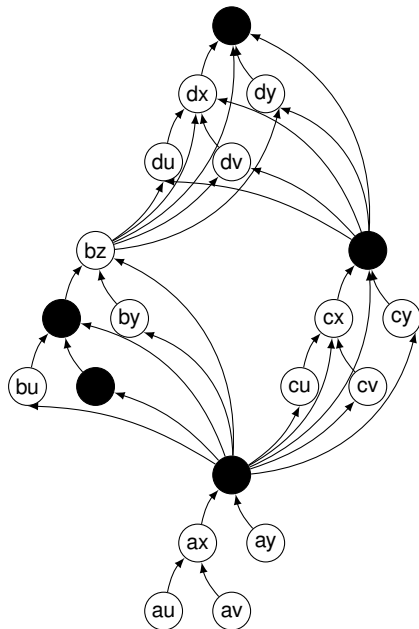
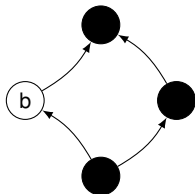
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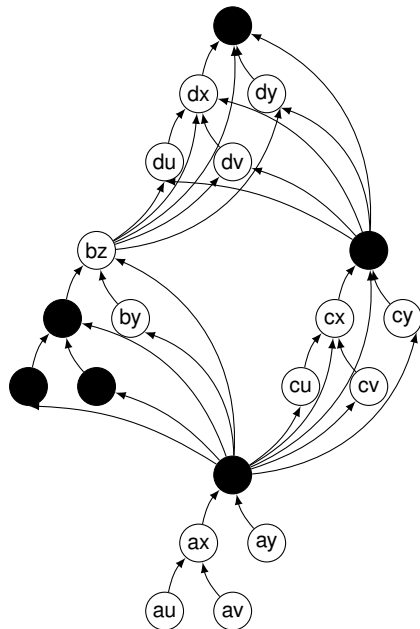
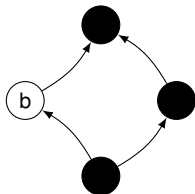
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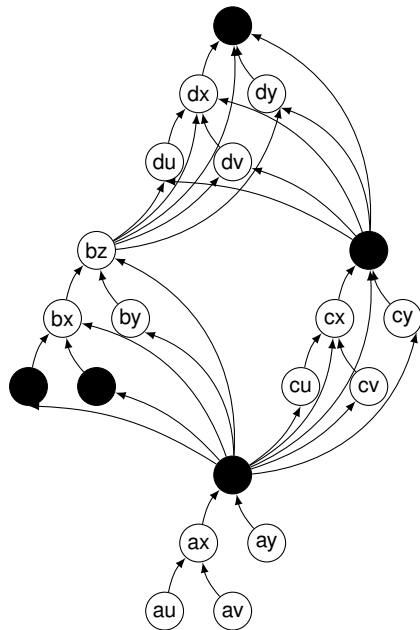
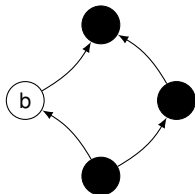
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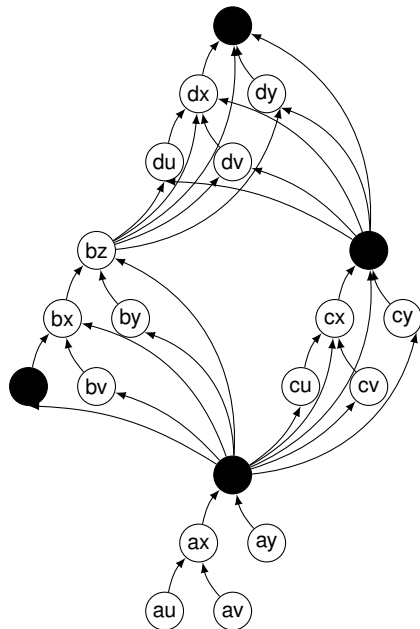
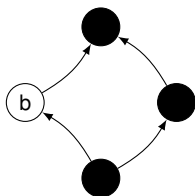
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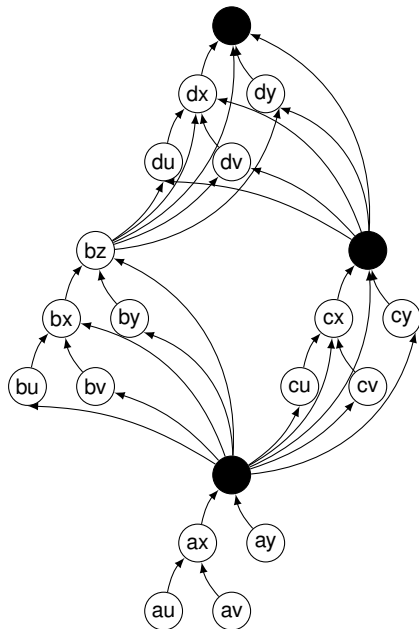
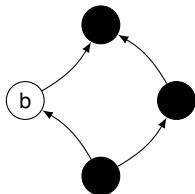
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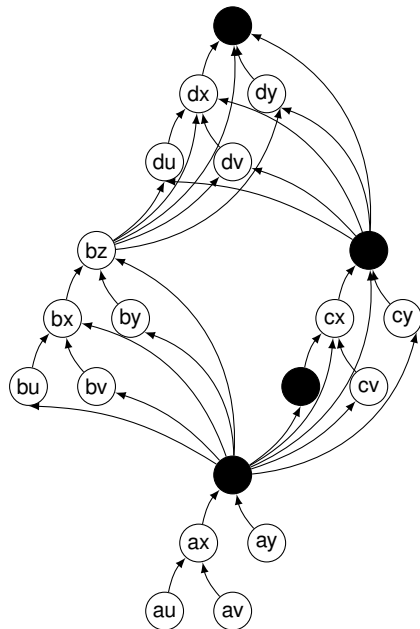
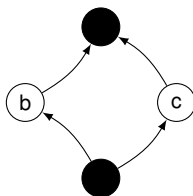
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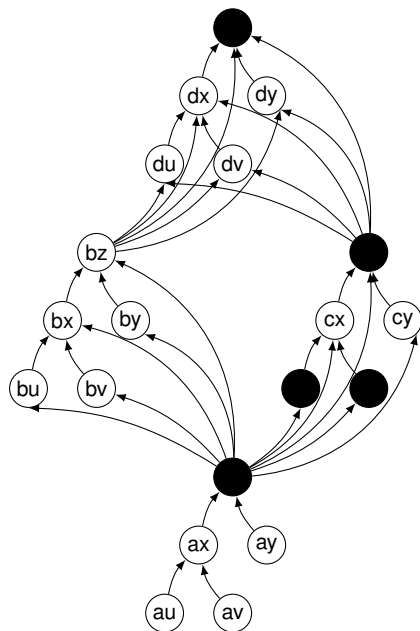
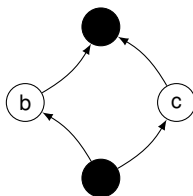
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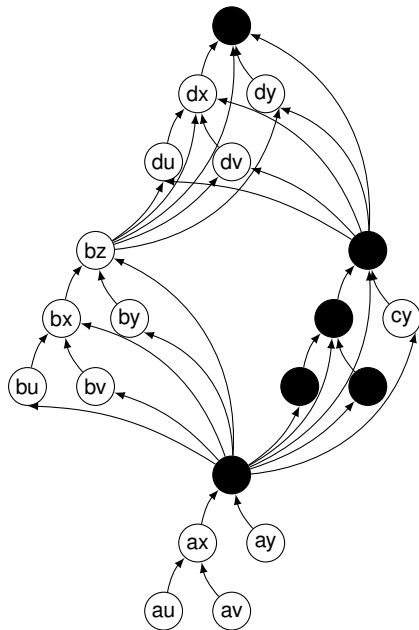
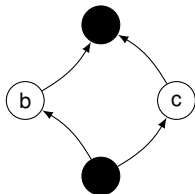
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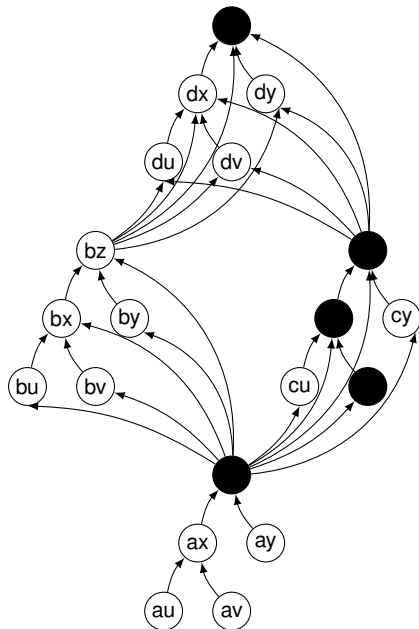
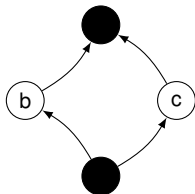
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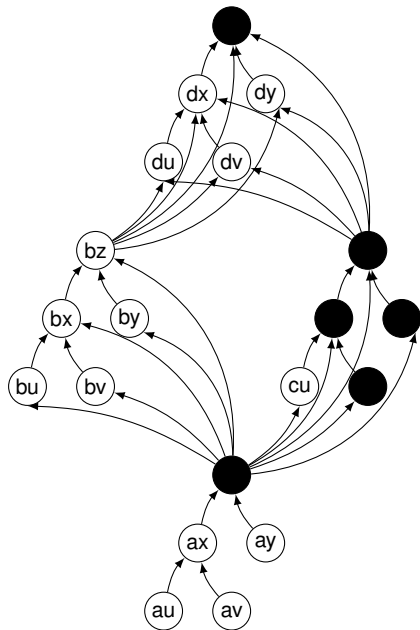
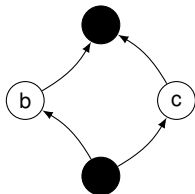
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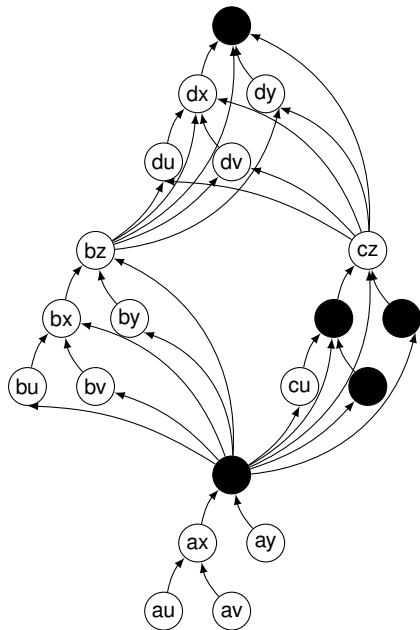
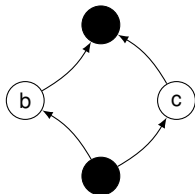
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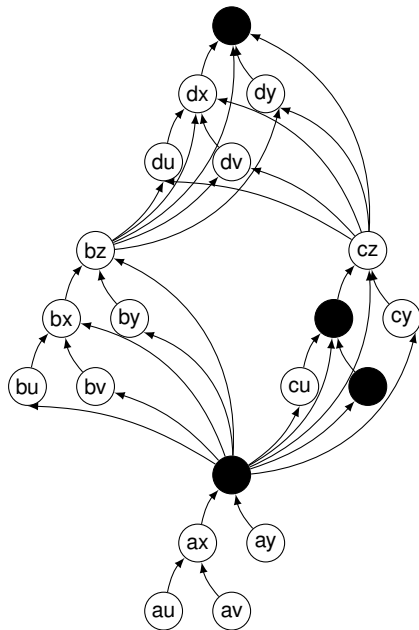
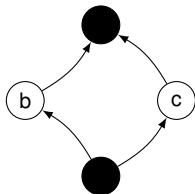
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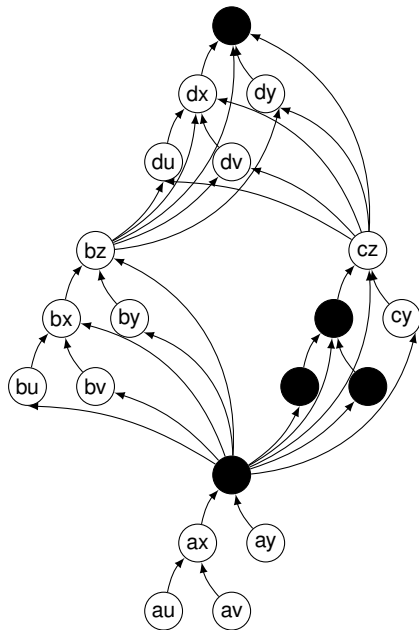
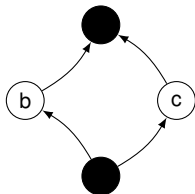
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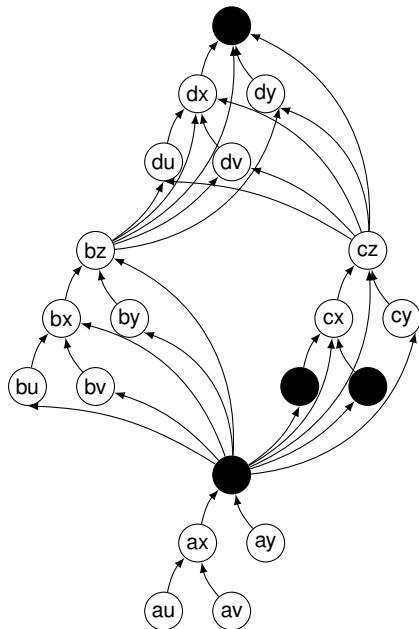
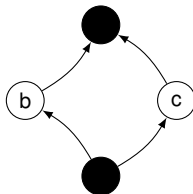
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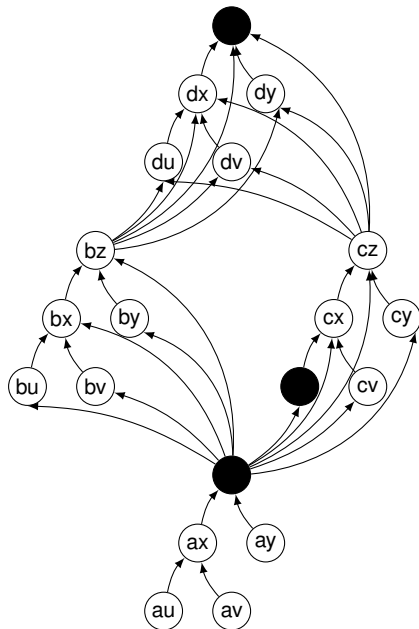
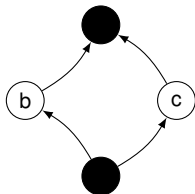
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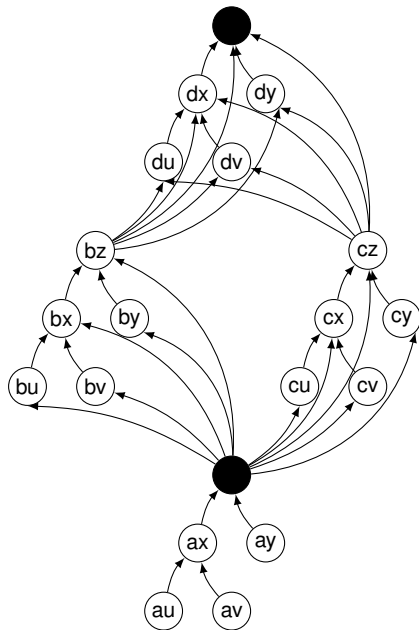
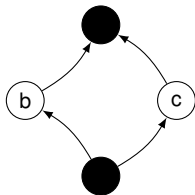
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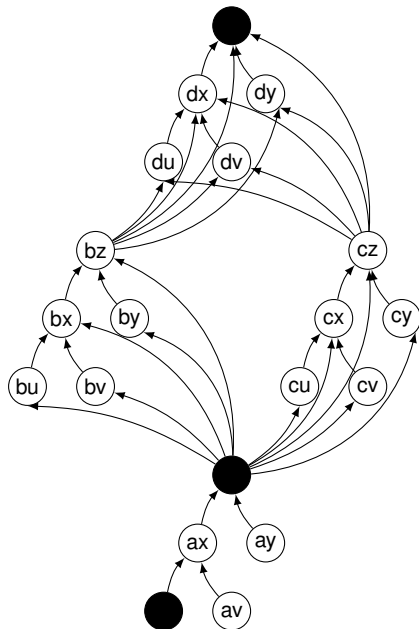
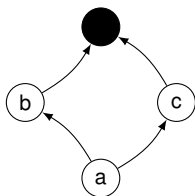
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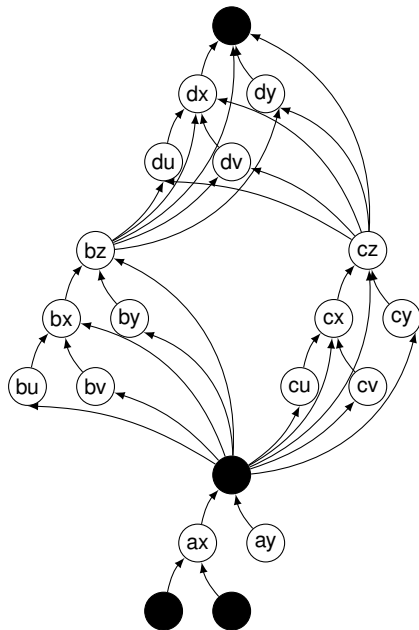
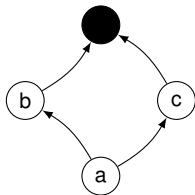
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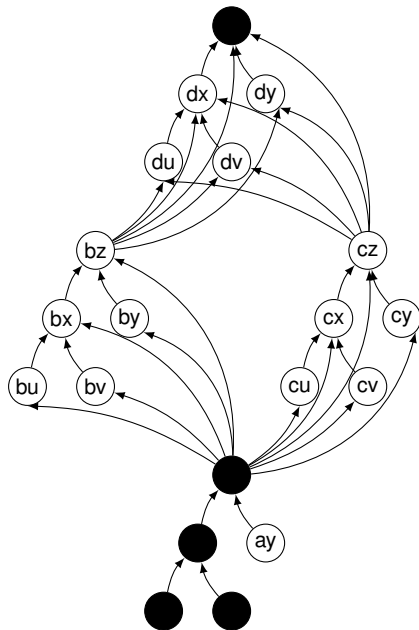
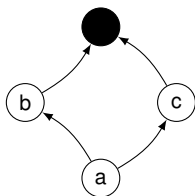
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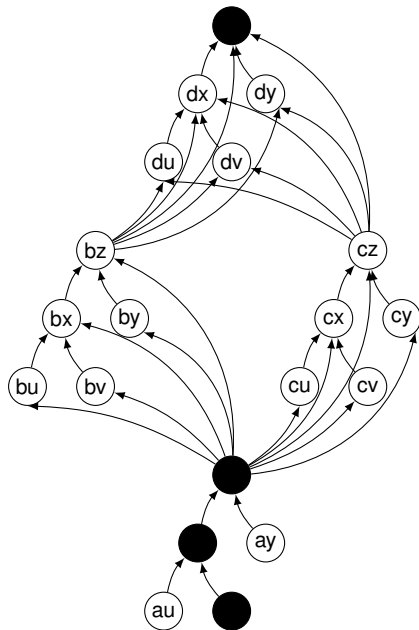
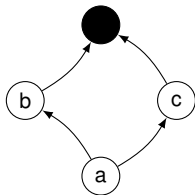
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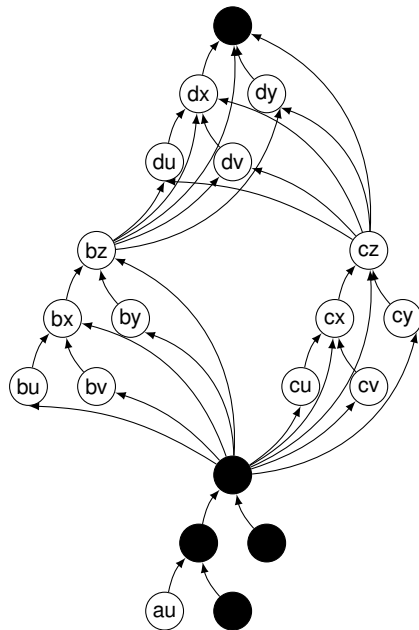
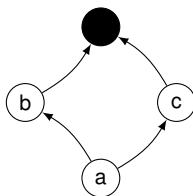
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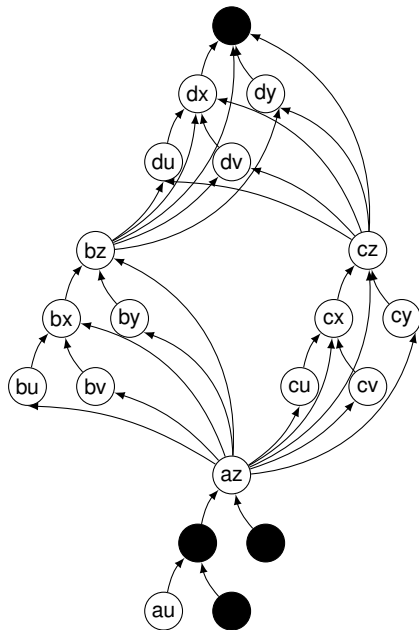
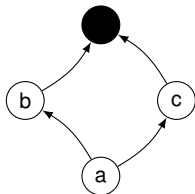
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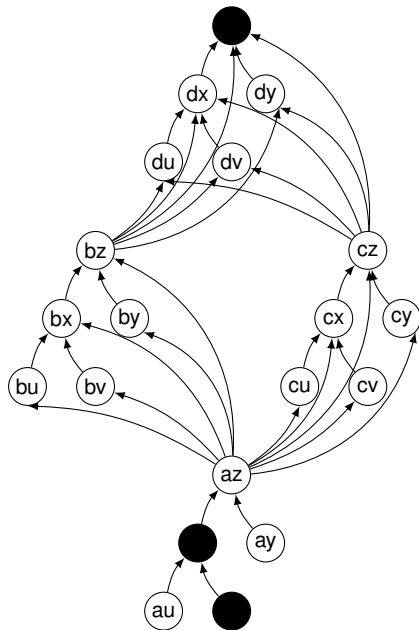
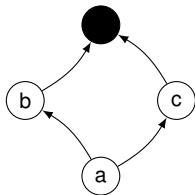
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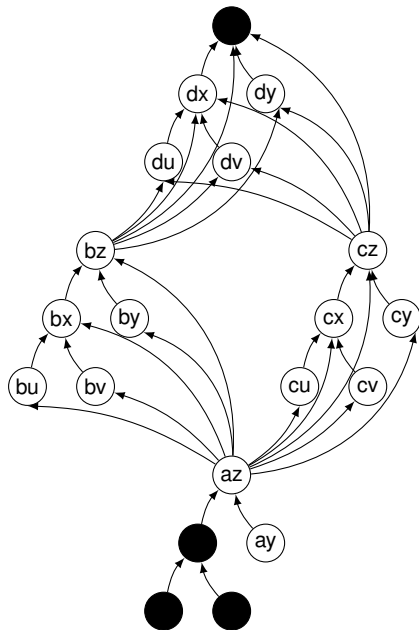
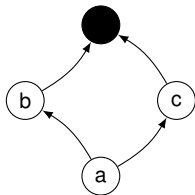
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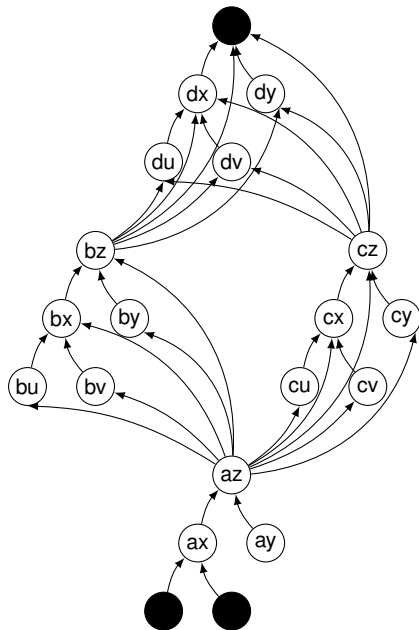
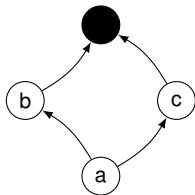
Upper Bound

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- ▶ Move \rightarrow pebbling of inner graph
- ▶ $\text{price}(O) + \text{price}(I) - 1$ pebbles



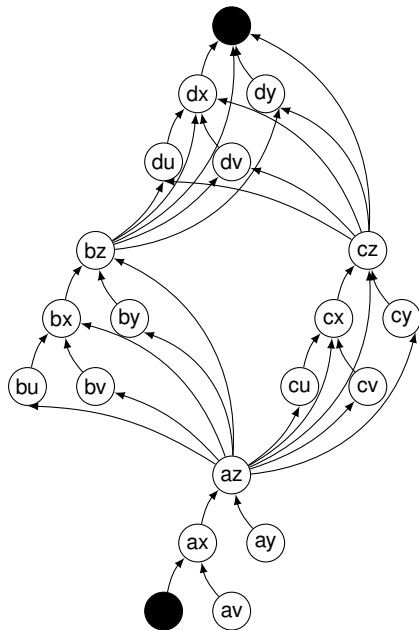
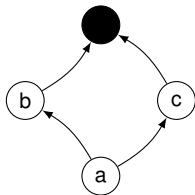
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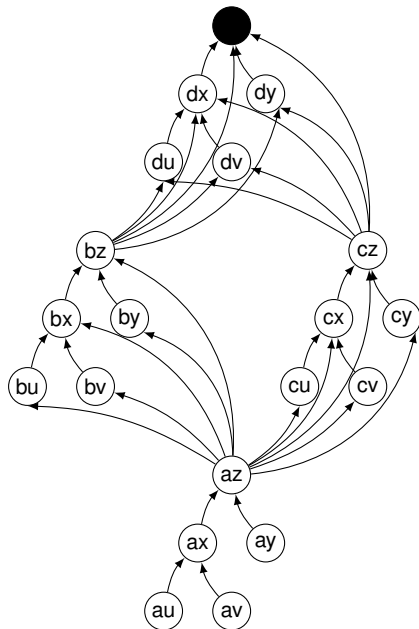
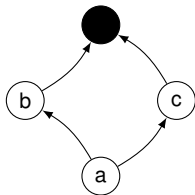
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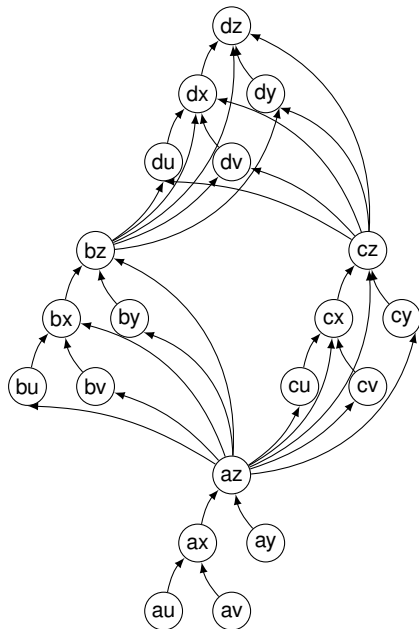
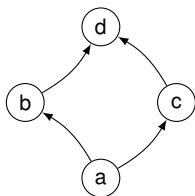
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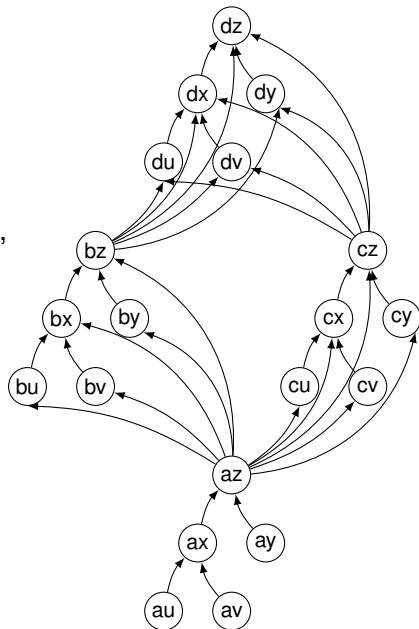
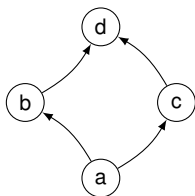
Lower Bound: first attempt

- ▶ Extract pebbling for outer graph



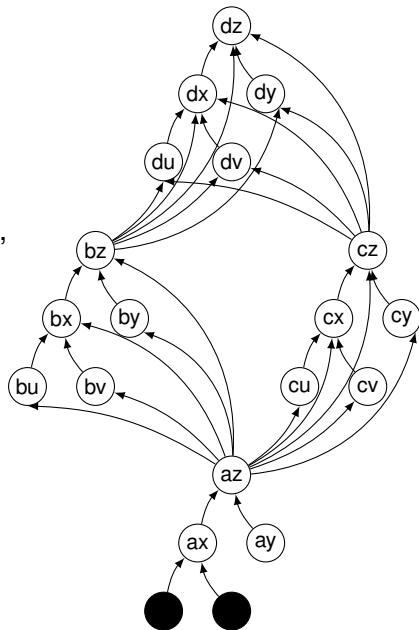
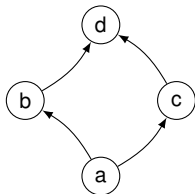
Lower Bound: first attempt

- ▶ Extract pebbling for outer graph
- ▶ Pebble on local sink \rightarrow pebble
- ▶ Want price(O) copies with a pebble, and one with price(I) pebbles



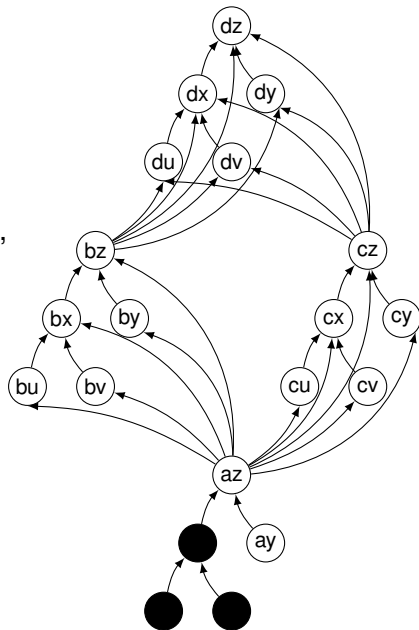
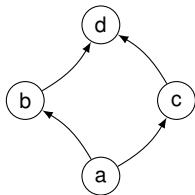
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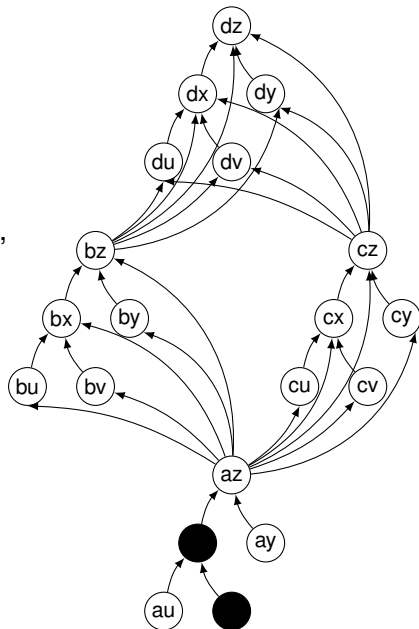
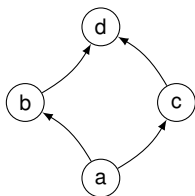
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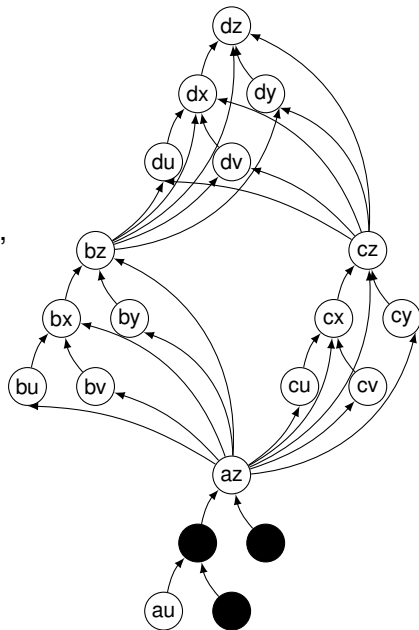
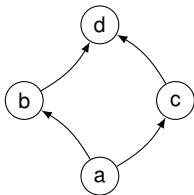
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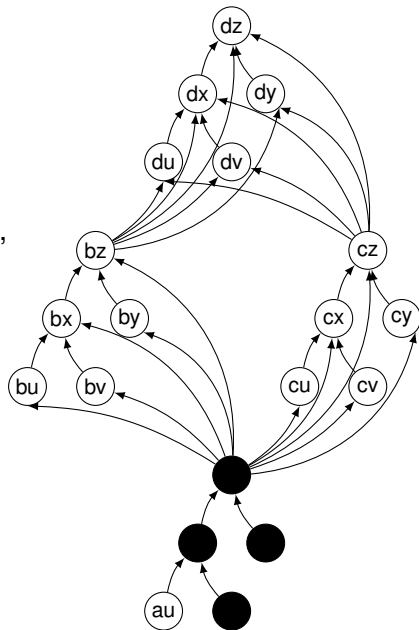
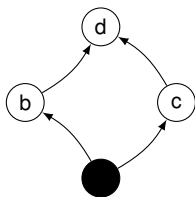
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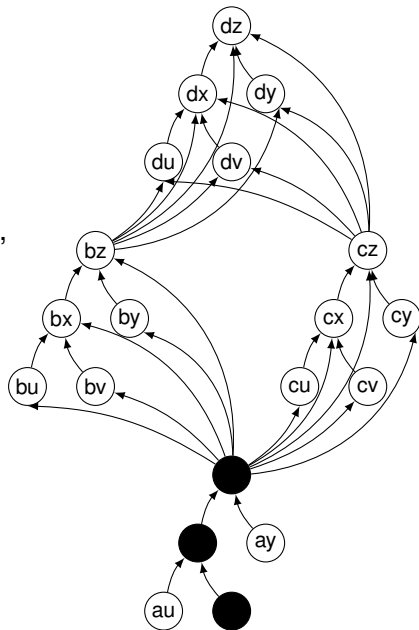
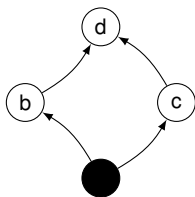
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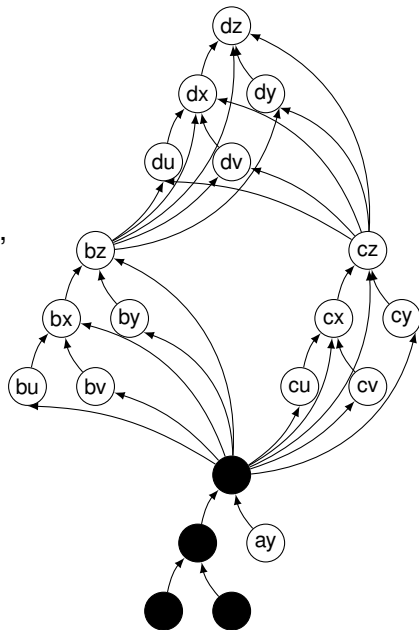
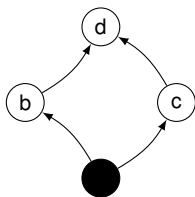
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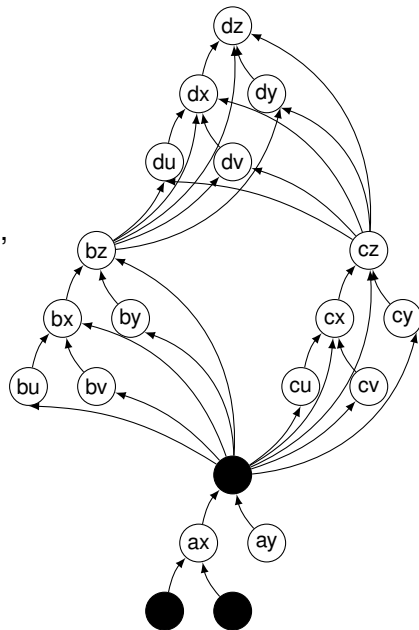
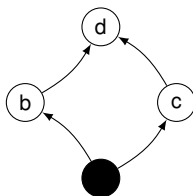
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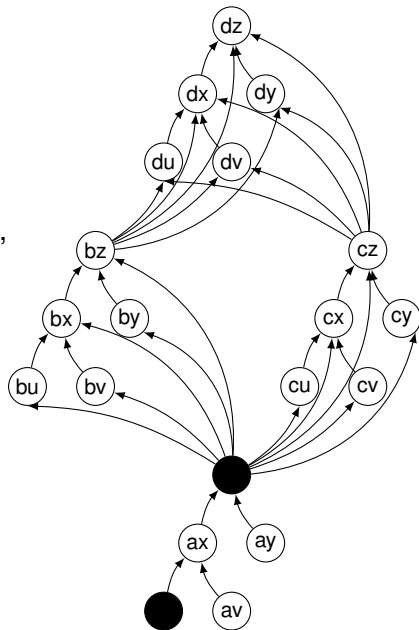
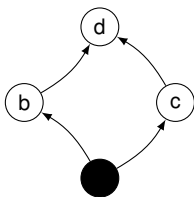
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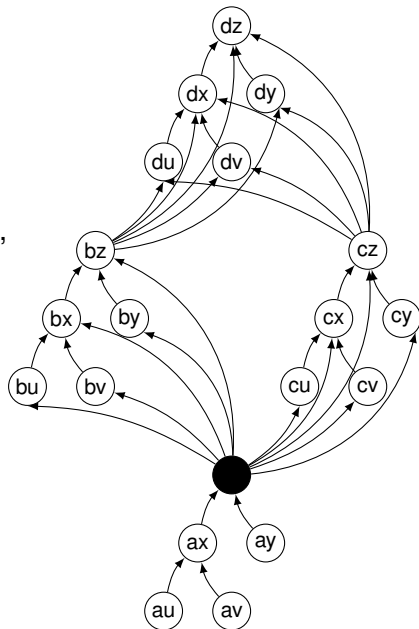
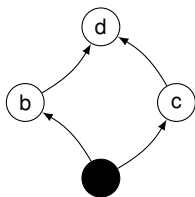
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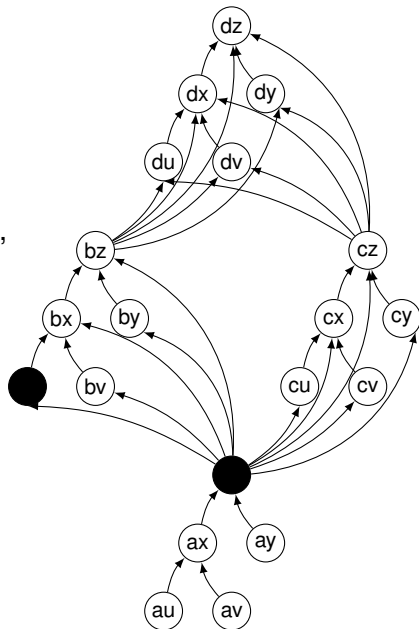
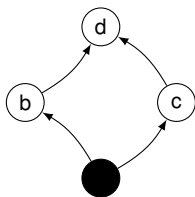
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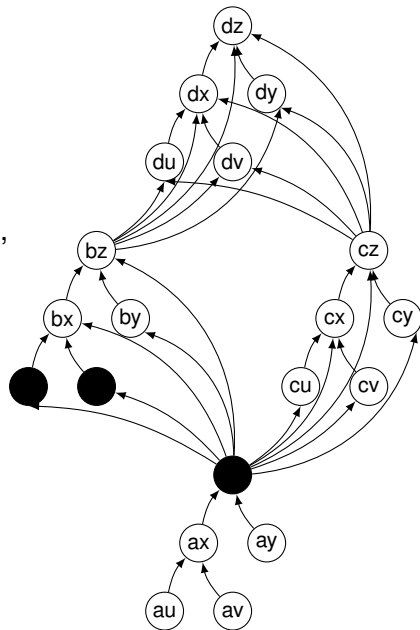
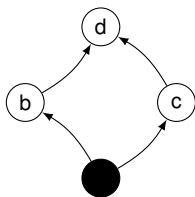
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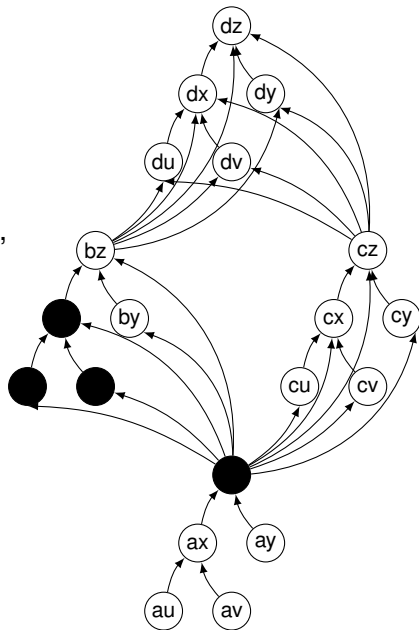
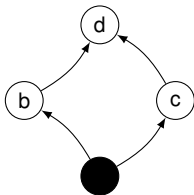
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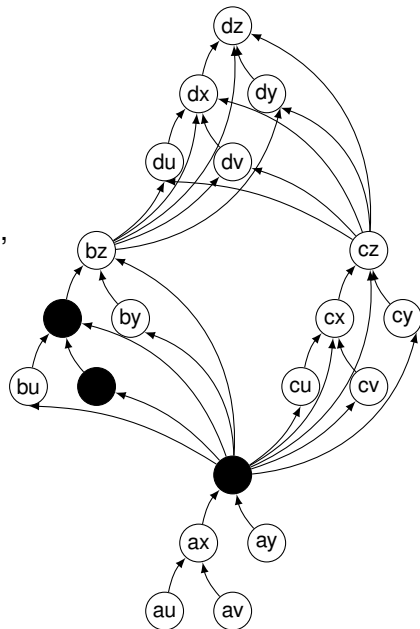
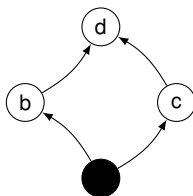
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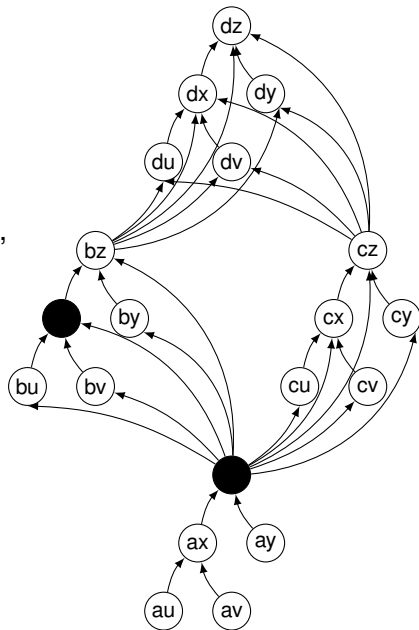
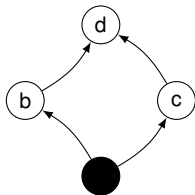
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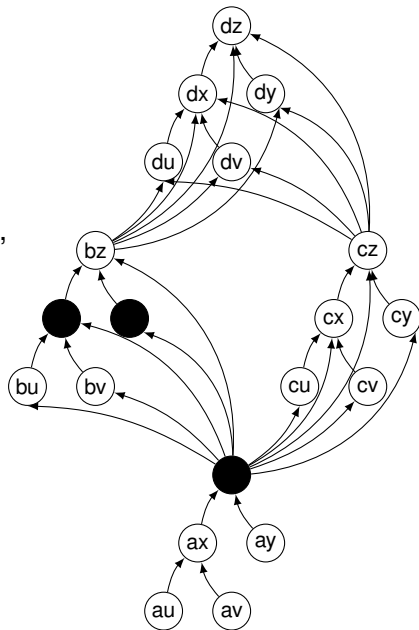
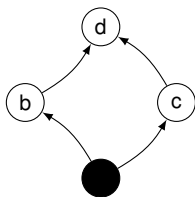
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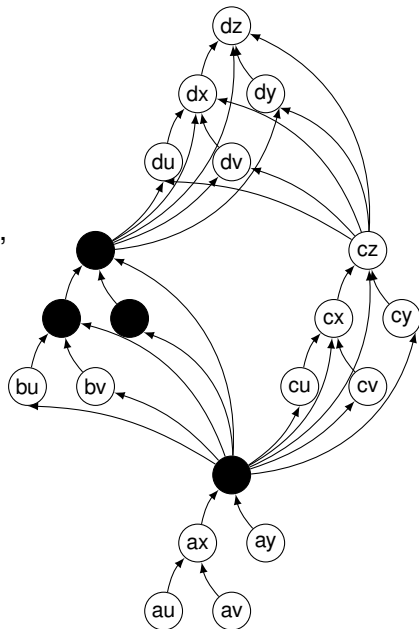
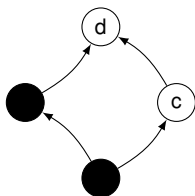
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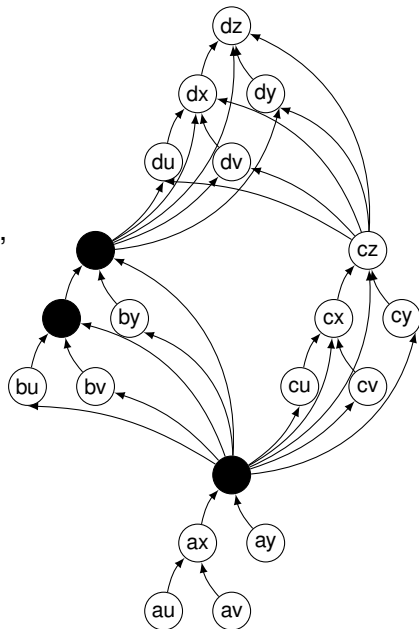
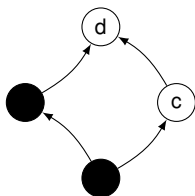
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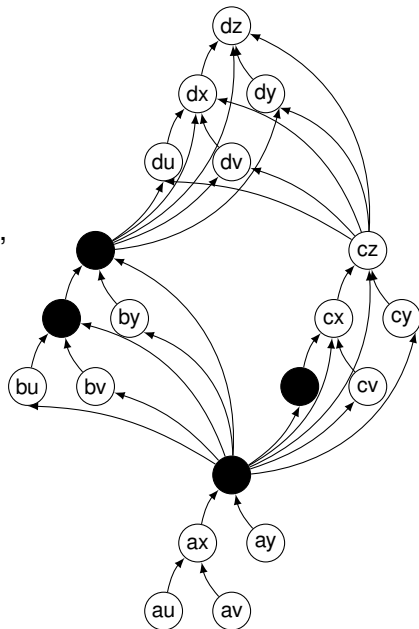
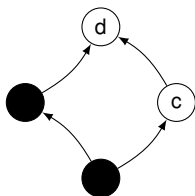
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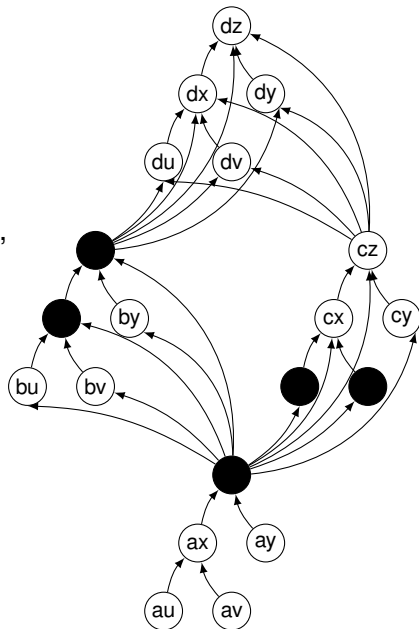
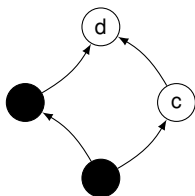
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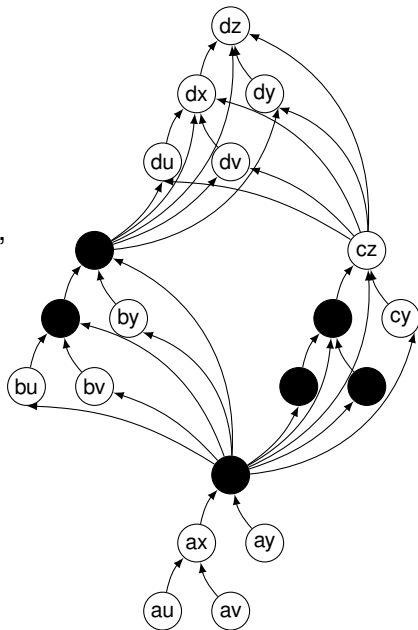
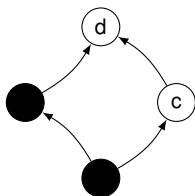
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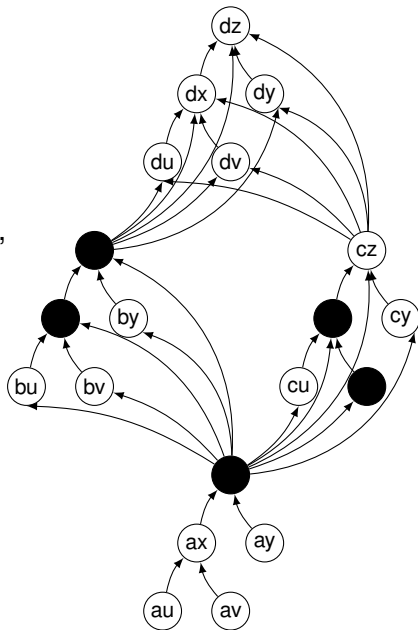
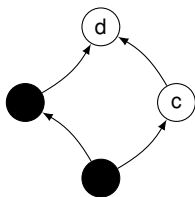
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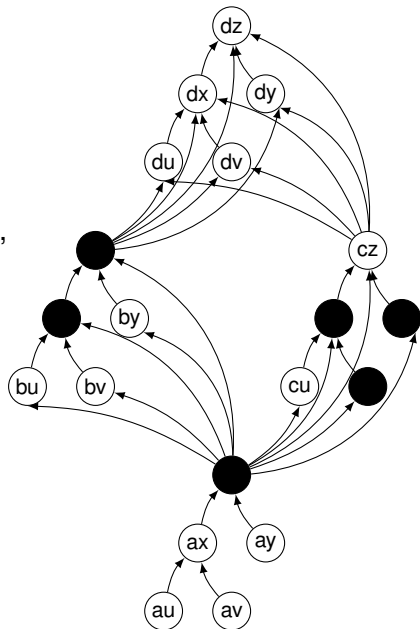
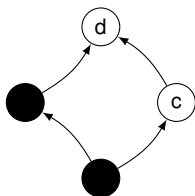
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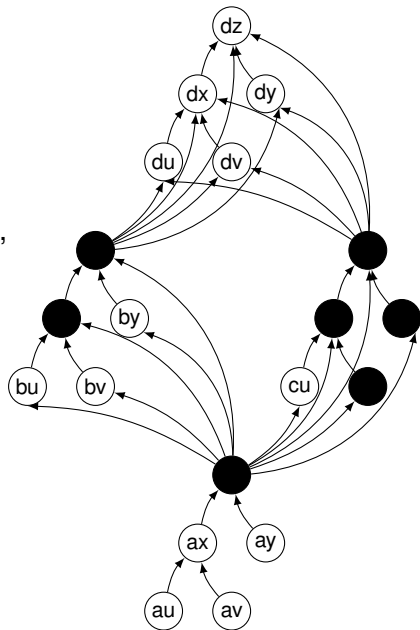
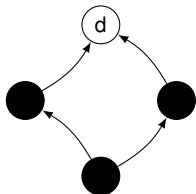
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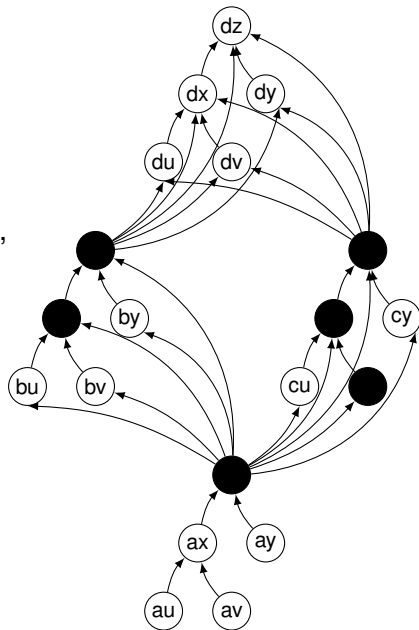
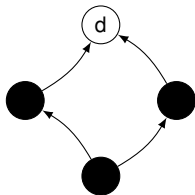
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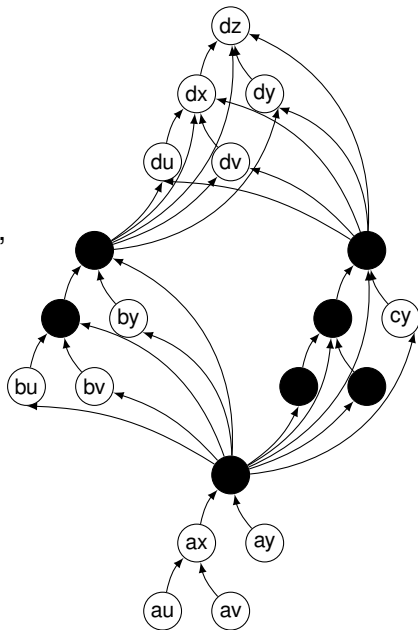
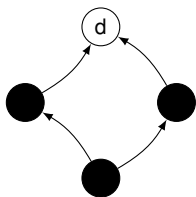
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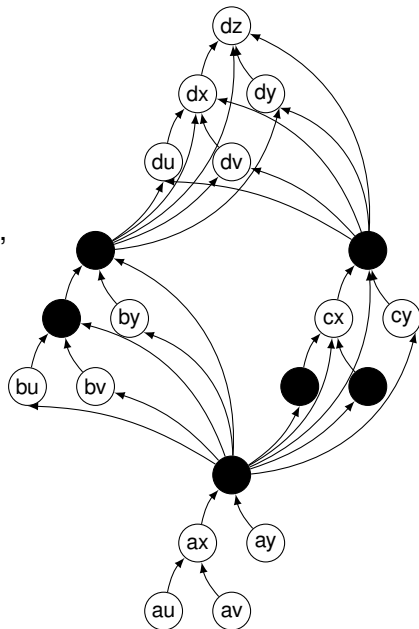
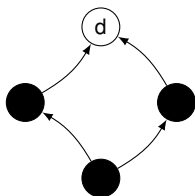
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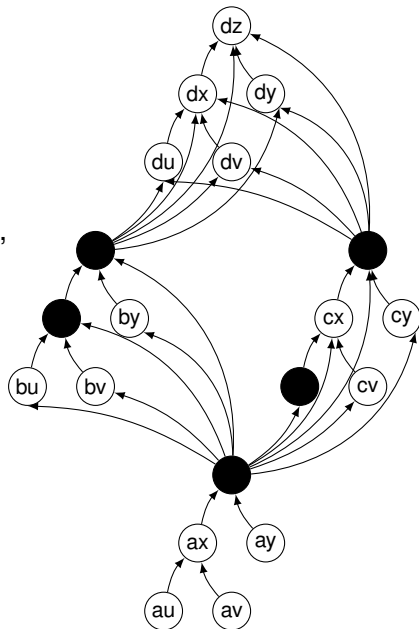
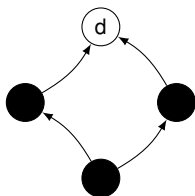
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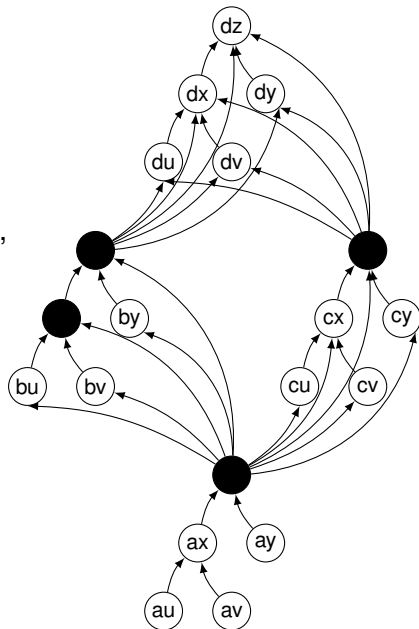
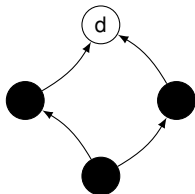
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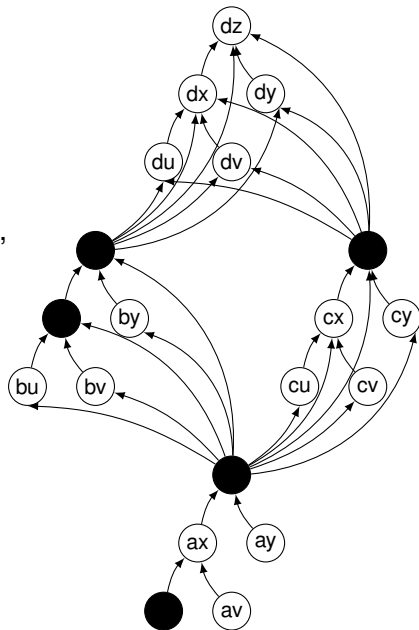
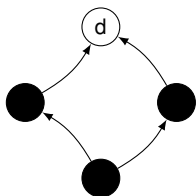
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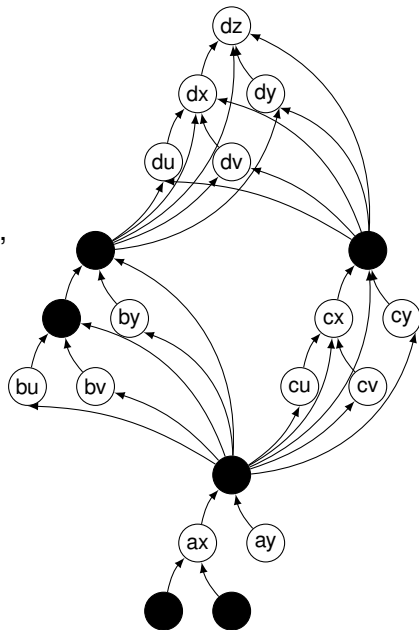
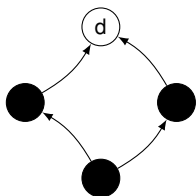
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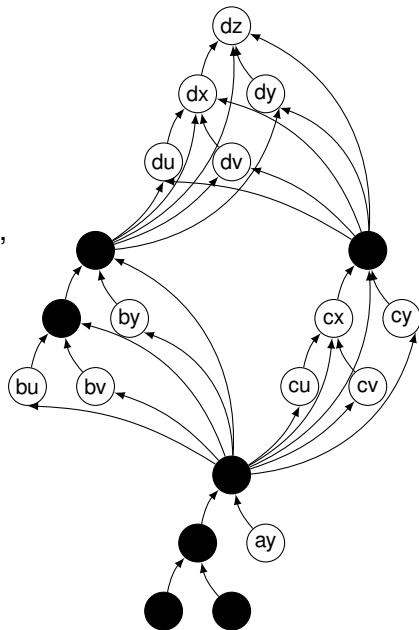
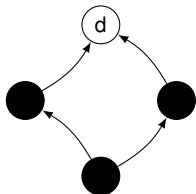
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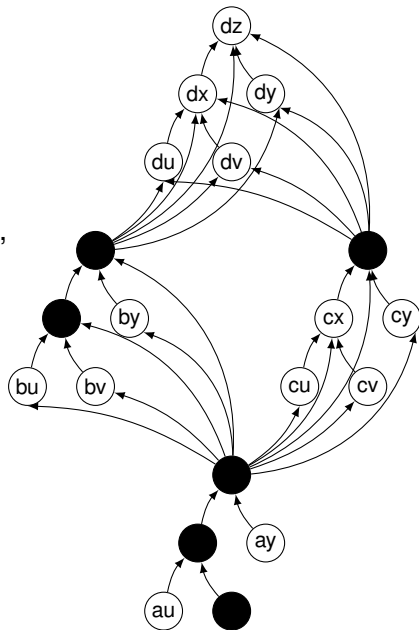
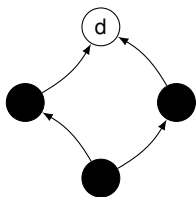
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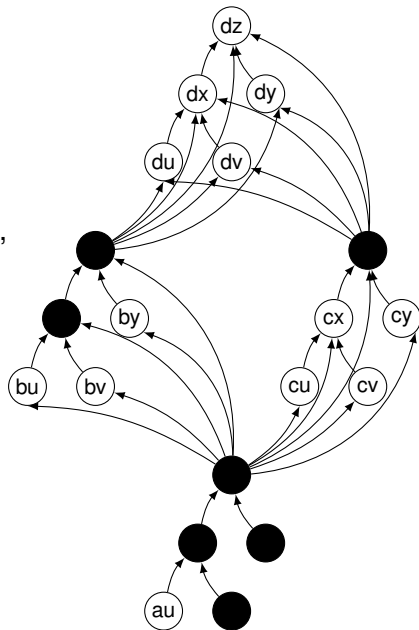
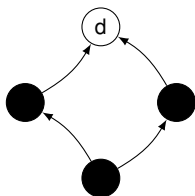
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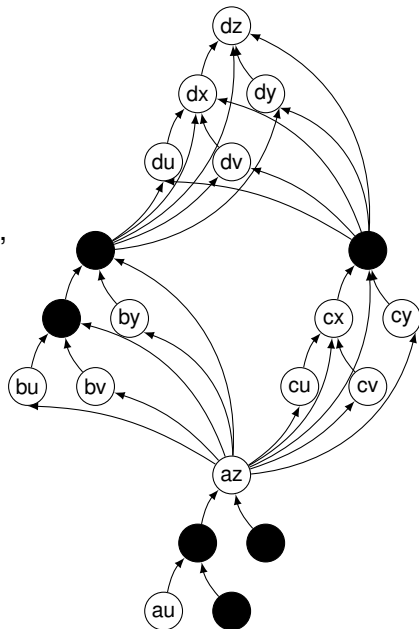
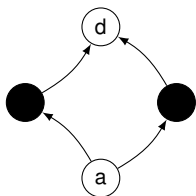
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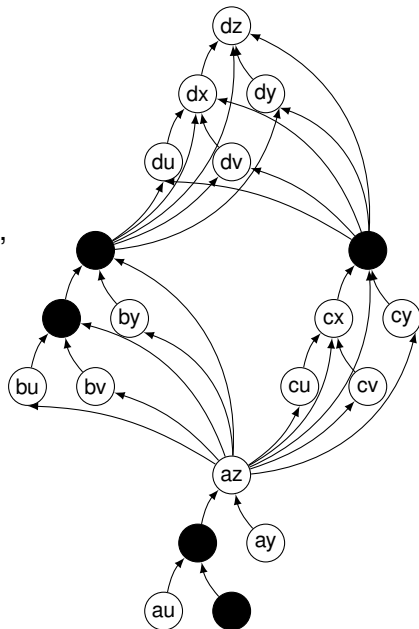
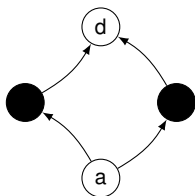
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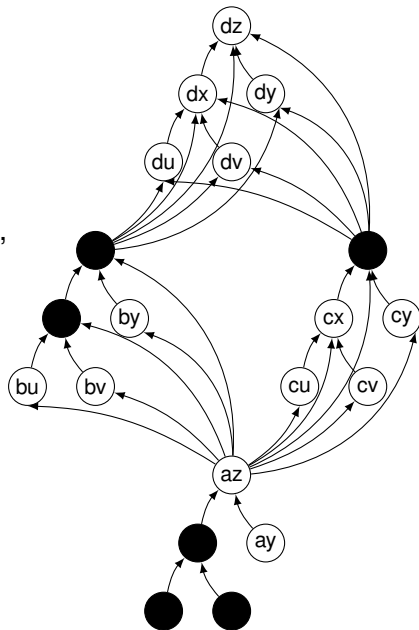
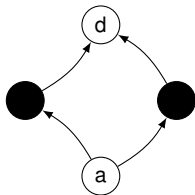
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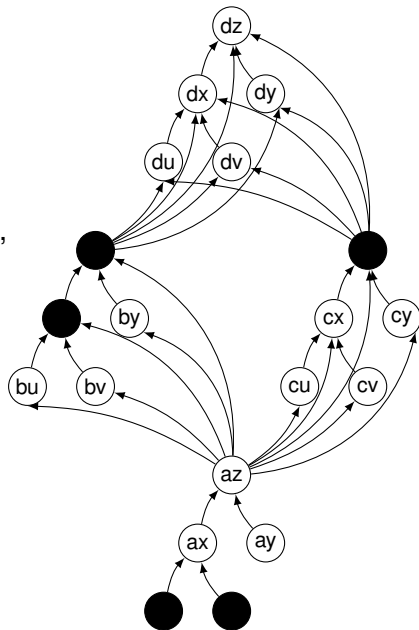
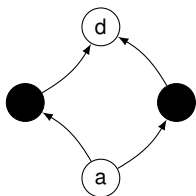
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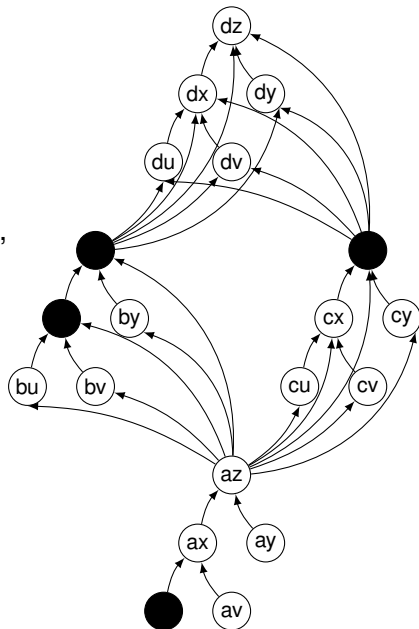
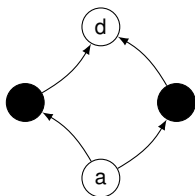
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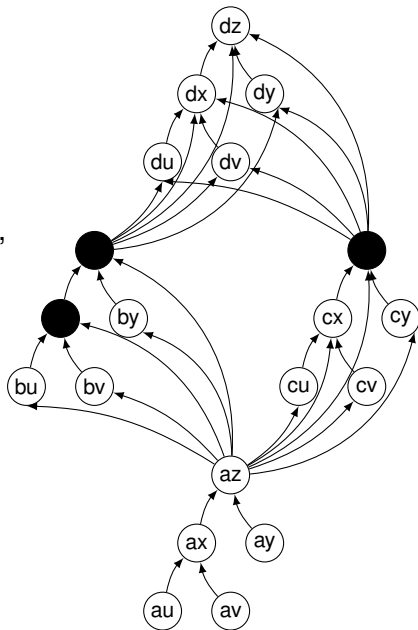
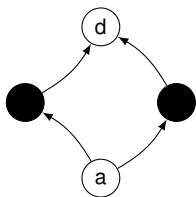
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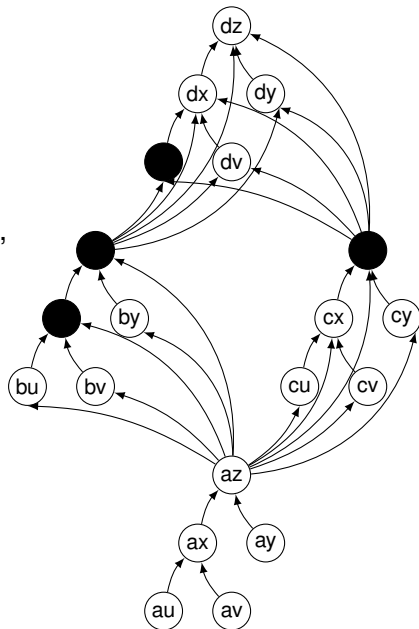
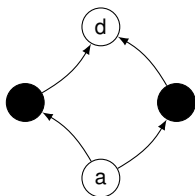
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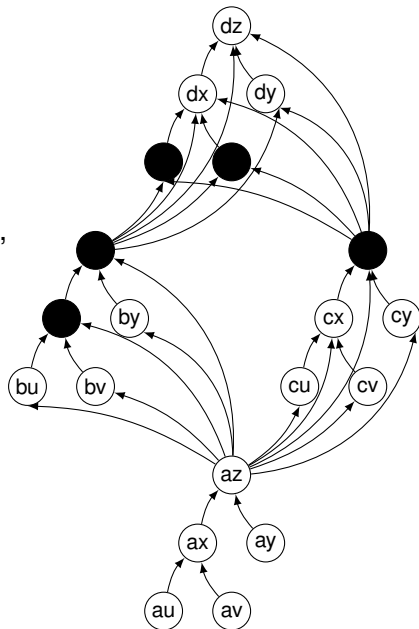
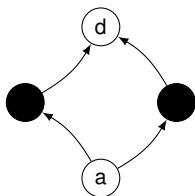
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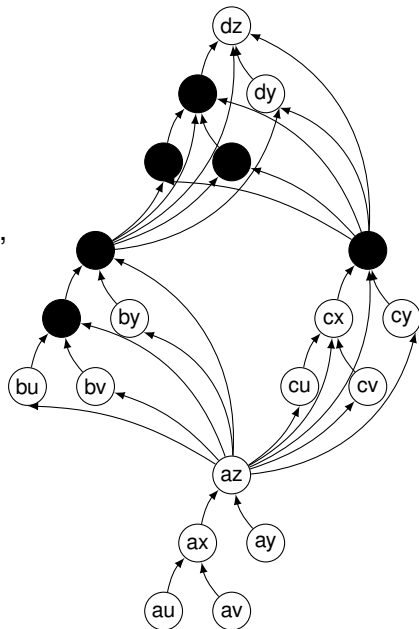
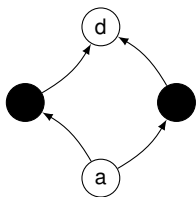
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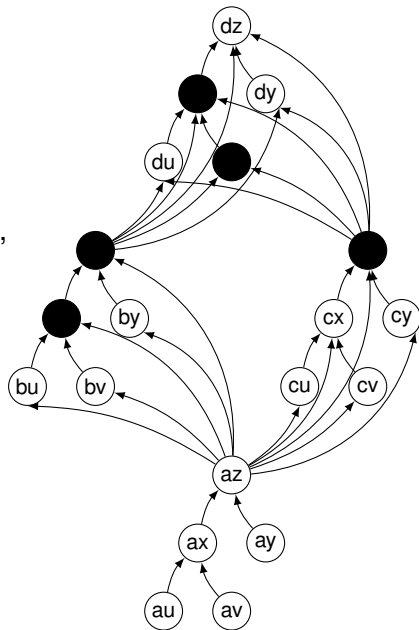
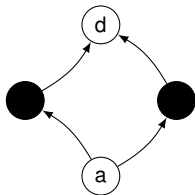
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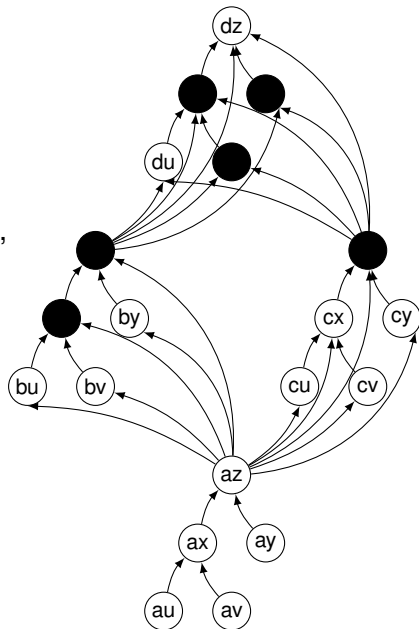
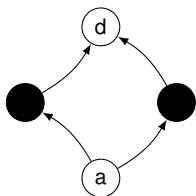
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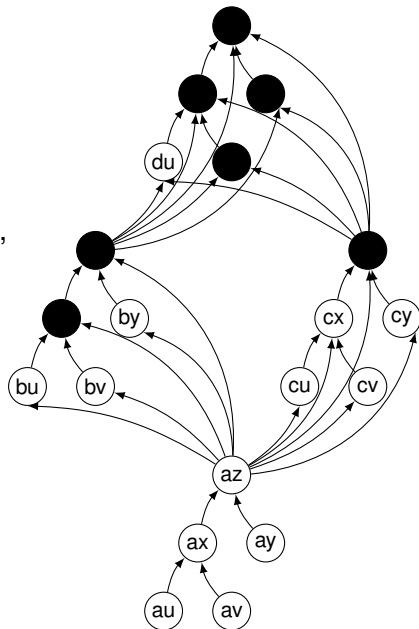
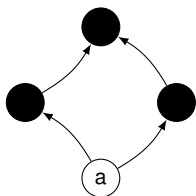
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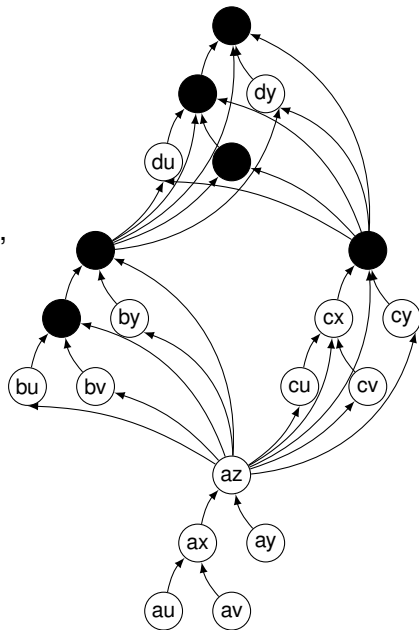
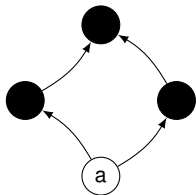
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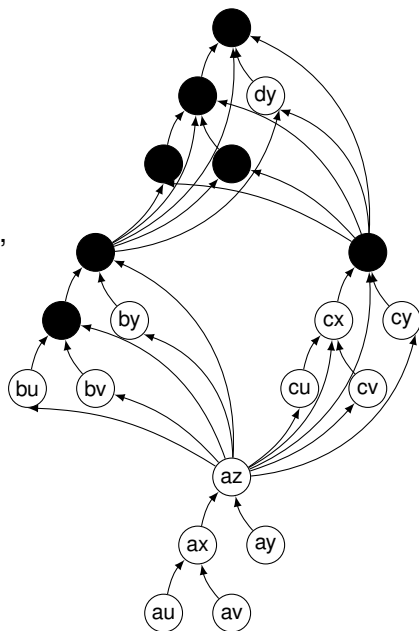
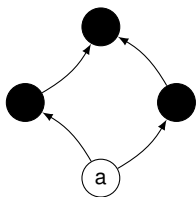
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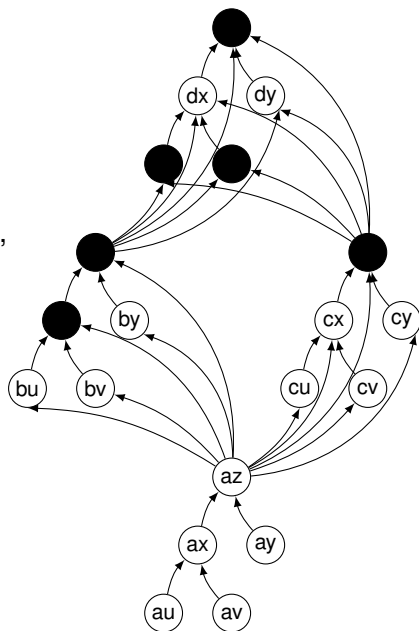
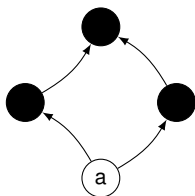
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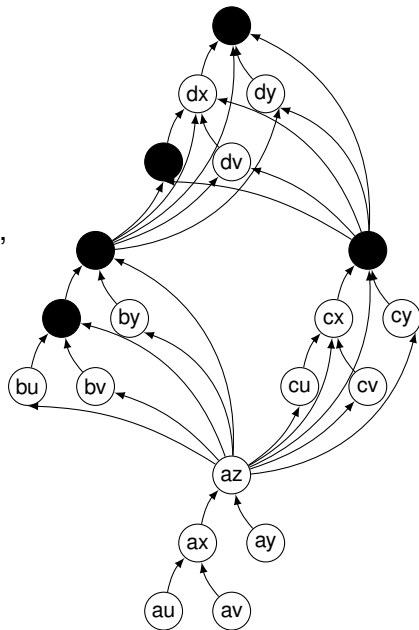
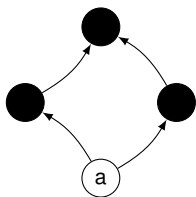
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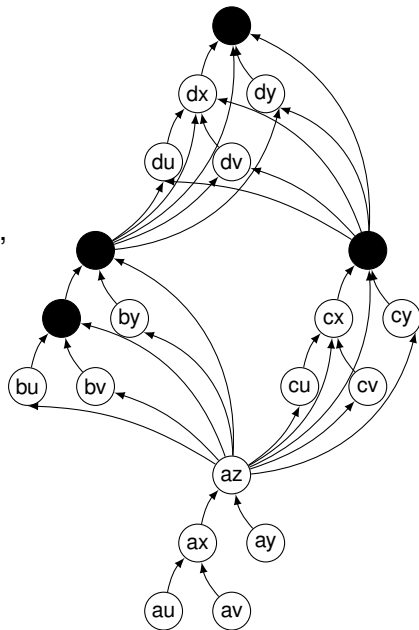
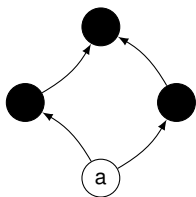
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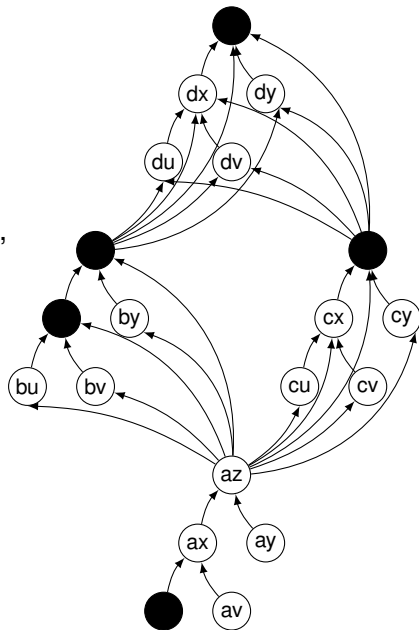
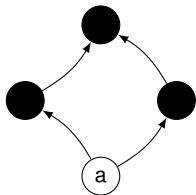
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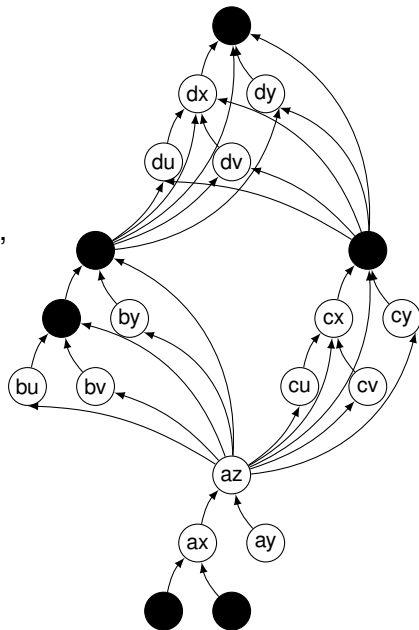
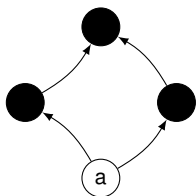
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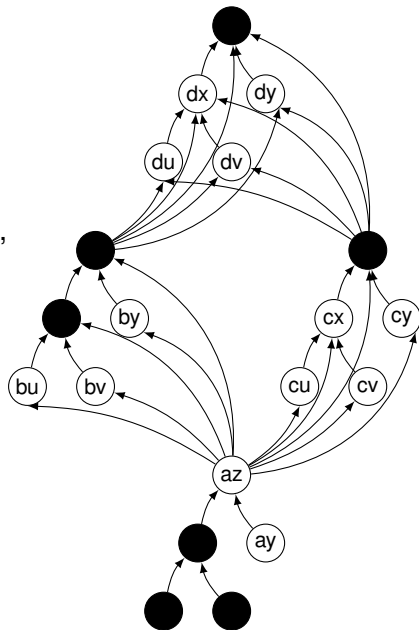
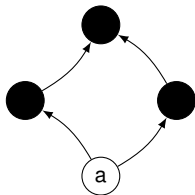
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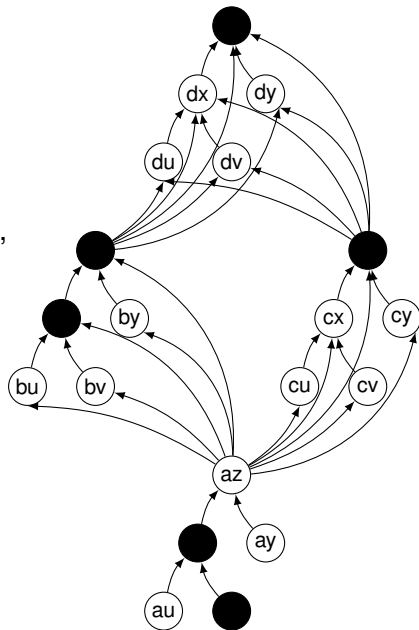
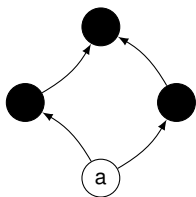
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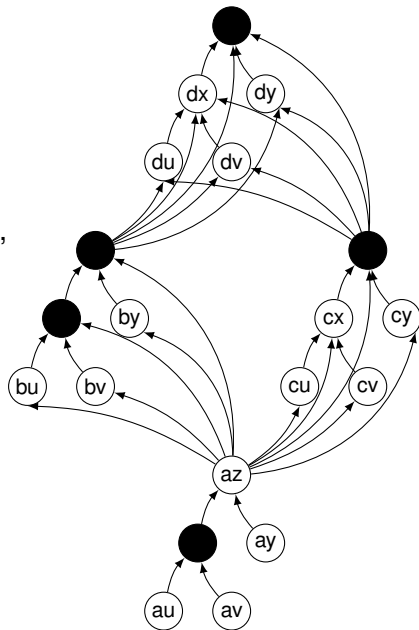
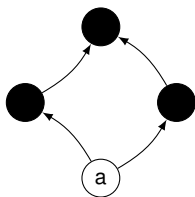
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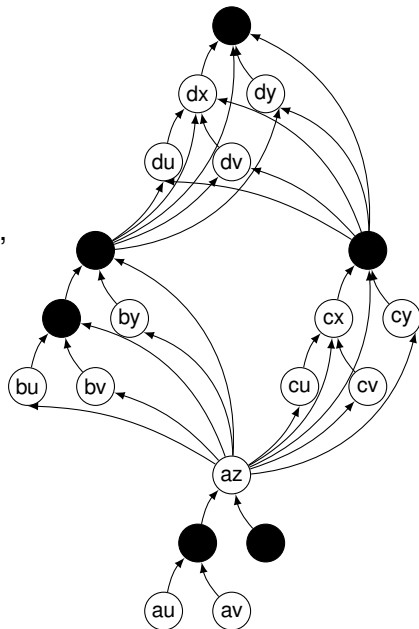
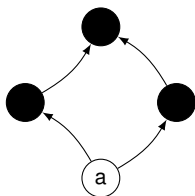
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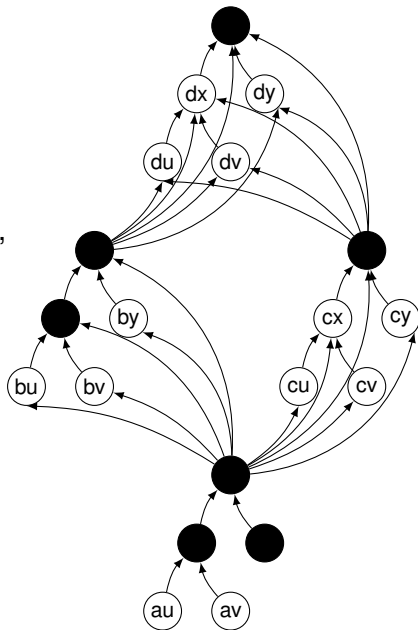
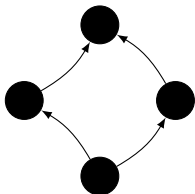
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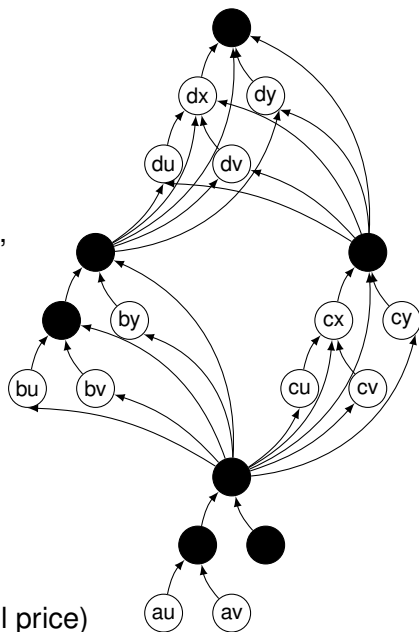
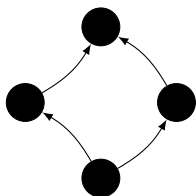
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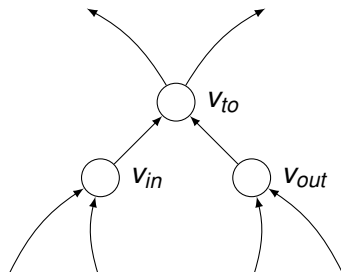
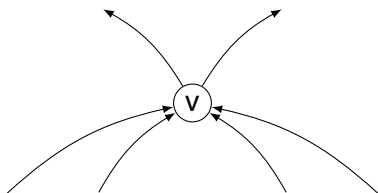
- ▶ Ignore sink, count price of configuration
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- ▶ It works!

Making the Indegree Constant

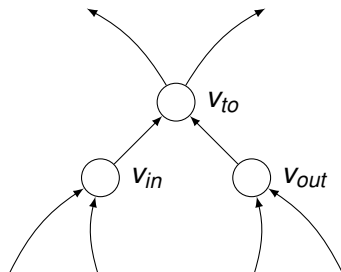
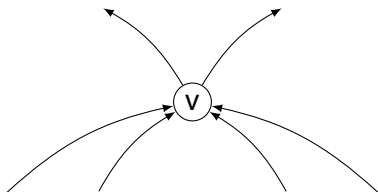
Replace v by v_{in} , v_{out} , v_{to}



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Thanks!