In Between Resolution and Cutting Planes
Proof Systems for Pseudo-Boolean SAT Solving

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The Power of CDCL Solvers

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- Replace heuristics by nondeterminism $\rightarrow$ CDCL proof system

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*(Ignoring preprocessing)

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And the opposite direction?

Theorem [Pipatsrisawat, Darwiche '09; Atserias, Fichte, Thurley '09]

CDCL $\equiv_{\text{poly}}$ Resolution

- CDCL can simulate any resolution proof
- Not true for DPLL: limited to tree-like

More Powerful Solvers

Resolution is a weak proof system

- e.g. cannot count
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Pseudo-Boolean constraints more expressive

$$x_1 + \cdots + x_n \geq n/2$$
$$\overline{x_1} + \cdots + \overline{x_n} \geq n/2$$

Build solvers with pseudo-Boolean constraints?

What do we do

Question

How powerful are pseudo-Boolean SAT solvers?

Study proof systems arising from pseudo-Boolean SAT solvers

Cutting Planes

All pseudo-Boolean proofs are cutting planes proofs

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Work with linear pseudo-Boolean inequalities

$$x \vee \overline{y} \quad \to \quad x + \overline{y} \geq 1 \quad \equiv \quad x + (1 - y) \geq 1$$

$\overline{y} = 1 - y$

degree

Cutting Planes

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$\overline{y} = 1 - y$ degree

Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (\alpha a_i + \beta b_i) x_i \geq \alpha a + \beta b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a/k \rceil}$$

Goal: derive $0 \geq 1$

Addition in Practice

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (\alpha a_i + \beta b_i) x_i \geq \alpha a + \beta b}$$

- Unbounded choices
- Need a reason to add inequalities

Division in Practice

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i/k) x_i \geq \lceil a/k \rceil}$$

- Too expensive

Weaker Rules

What is the bare minimum to simulate resolution?

$$\frac{x \lor y \lor \overline{z} \qquad \overline{x} \lor y}{y \lor \overline{z}}$$

Weaker Rules

What is the bare minimum to simulate resolution?

$$\frac{x \vee y \vee \overline{z} \qquad \overline{x} \vee y}{y \vee \overline{z}} \qquad\qquad \frac{x + y + \overline{z} \geq 1 \qquad \overline{x} + y \geq 1}{x + \overline{x} + 2y + \overline{z} \geq 2}$$

Weaker Rules

What is the bare minimum to simulate resolution?

$$\frac{x \vee y \vee \overline{z} \qquad \overline{x} \vee y}{y \vee \overline{z}}$$

$$\frac{x + y + \overline{z} \geq 1 \qquad \overline{x} + y \geq 1}{\text{✻} + 2y + \overline{z} \geq 1}$$

- Addition only if some variable cancels

Weaker Rules

What is the bare minimum to simulate resolution?

$$\frac{x \vee y \vee \bar{z} \qquad \bar{x} \vee y}{y \vee \bar{z}}$$

$$\dfrac{\dfrac{x + y + \bar{z} \geq 1 \qquad \bar{x} + y \geq 1}{2y + \bar{z} \geq 1}}{y + \bar{z} \geq 1}$$

- Addition only if some variable cancels
- Division brings coefficients down to degree

Addition in Practice

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum(\alpha a_i + \beta b_i) x_i \geq \alpha a + \beta b}$$

- Unbounded choices
- Need a reason to add inequalities

Cancelling Addition

Some variable cancels: $\alpha a_i + \beta b_i = 0$

Division in Practice

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i/k) x_i \geq \lceil a/k \rceil}$$

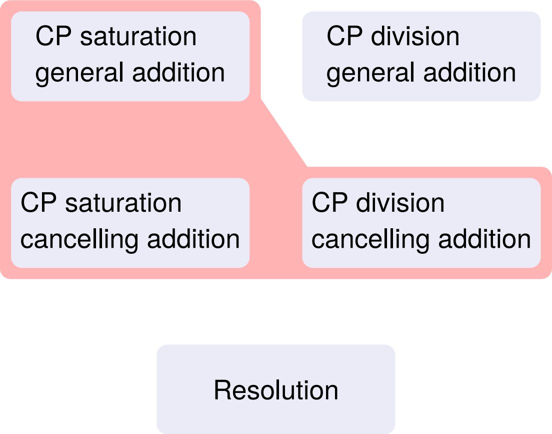
- Too expensive

Saturation

$$\frac{\sum a_i x_i \geq a}{\sum \min(a, a_i) x_i \geq a}$$

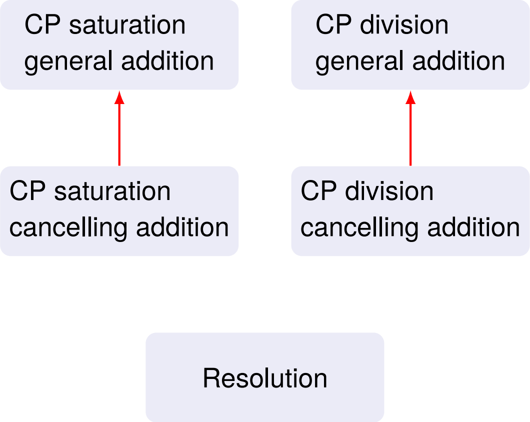
- Can simulate with repeated division

Proof Systems



Power of subsystems of CP?

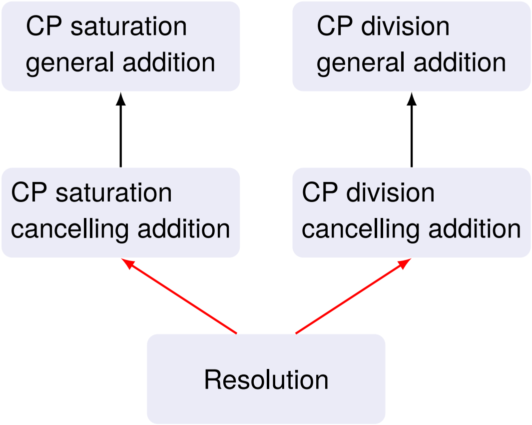
Proof Systems



Cancelling addition is a particular case of addition

$A \longrightarrow B$: B simulates A (with only polynomial loss)

Proof Systems

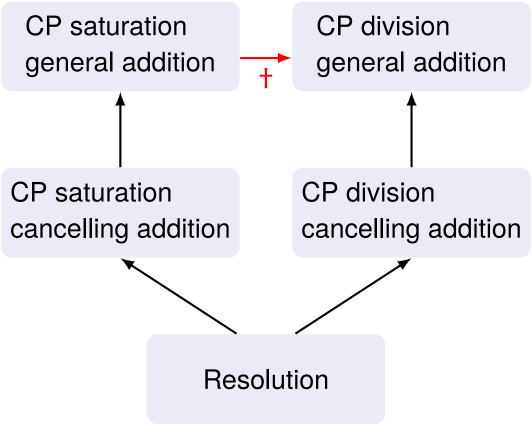


All subsystems simulate resolution

- Trivial over CNF inputs
- Also holds over linear pseudo-Boolean inputs

$A \longrightarrow B$: B simulates A (with only polynomial loss)

Proof Systems



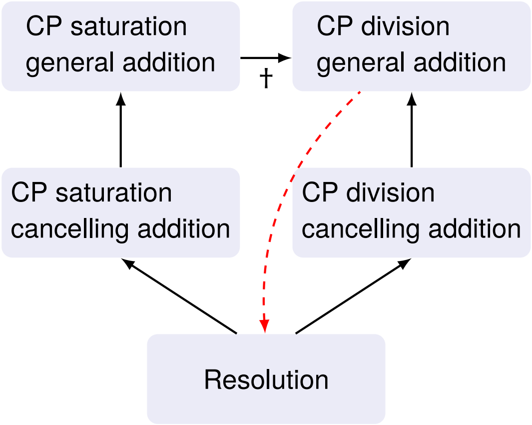
Repeated divisions simulate saturation

- Polynomial simulation only if polynomial coefficients

$A \longrightarrow B$: B simulates A (with only polynomial loss)

†: known only for polynomial-size coefficients

Proof Systems



CP stronger than resolution

- Pigeonhole principle
- Subset cardinality

have proofs of size

- polynomial in CP
- exponential in resolution

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Bad News

Theorem

On CNF inputs all subsystems as weak as resolution

- No subsystem is implicationally complete
- Solvers very sensitive to input encoding

Cancelling Addition $\equiv$ Resolution

Observation [Hooker '88]

Over CNF inputs CP with cancelling addition $\equiv$ resolution.

Cancelling Addition ≡ Resolution

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Over CNF inputs CP with cancelling addition ≡ resolution.

Proof Sketch

- Start with clauses (degree 1)
- Add two clauses $\to$ a clause

$$\frac{x + \sum y_i \geq 1 \qquad \overline{x} + \sum y_i \geq 1}{\ast + 1 + \sum y_i \geq 1 + 1}$$

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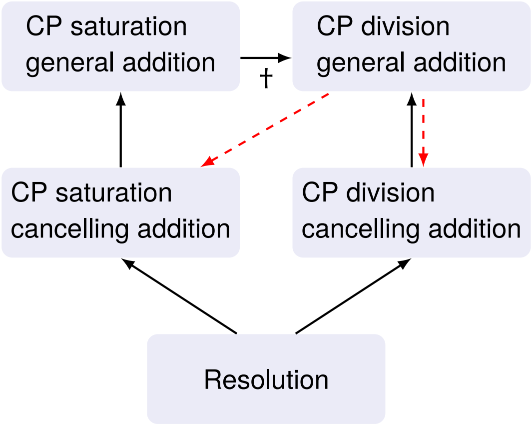
Proof Sketch

- Start with clauses (degree 1)
- Add two clauses $\rightarrow$ a clause

$$\frac{x + \sum y_i \geq 1 \qquad \overline{x} + \sum y_i \geq 1}{\sum y_i \geq 1} \equiv \frac{x \vee C \qquad \overline{x} \vee D}{C \vee D}$$

Proof Systems



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Cancellation $\equiv$ Resolution

- Over CNF inputs

[Hooker '88]

- Pigeonhole principle
- Subset cardinality

have proofs of size

- polynomial in CP
- exponential in CP with cancelling addition and any rounding

Saturation $\equiv$ Resolution

Theorem

Over CNF inputs CP with saturation and polynomial coefficients $\equiv$ resolution.

Saturation $\equiv$ Resolution

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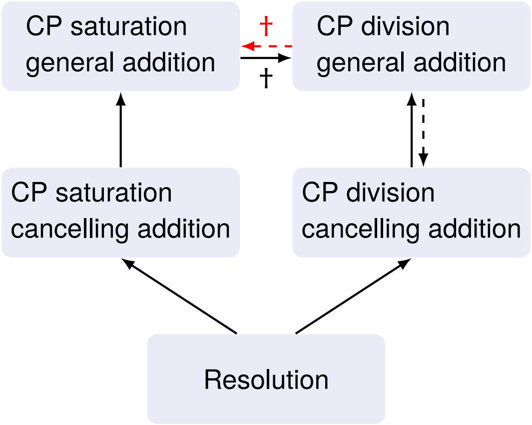
Over CNF inputs CP with saturation and polynomial coefficients $\equiv$ resolution.

Proof Sketch

- Represent inequality of degree A with A clauses
  - $x + 2y + \bar{z} \geq 2$ implied by $\{x \vee y,\ y \vee \bar{z}\}$
- Simulate addition step with A^2 resolution steps
- Saturation happens automatically

Proof Systems



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Saturation $\equiv$ Resolution

- Over CNF inputs

- Pigeonhole principle
- Subset cardinality

have proofs of size

- polynomial in CP
- exponential in CP with general addition and saturation

Easy Formulas

Pseudo-Boolean solvers $\equiv$ CP? No

Question

PB solvers $\equiv$ CP with cancelling addition and saturation?

Easy Formulas

Pseudo-Boolean solvers $\equiv$ CP? No

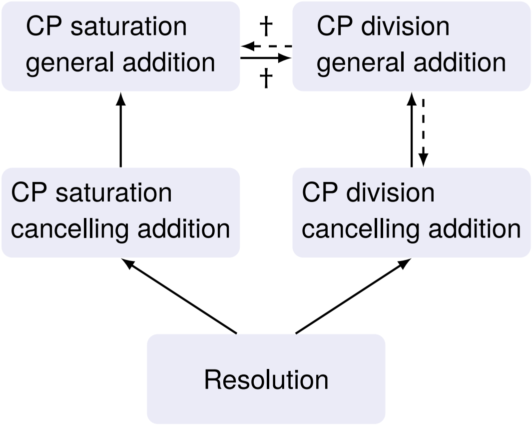
Question

PB solvers $\equiv$ CP with cancelling addition and saturation?

Craft combinatorial formulas easy for CP with cancelling addition and saturation

- All formulas without rational solutions
- Easy versions of NP-hard problems

Proof Systems



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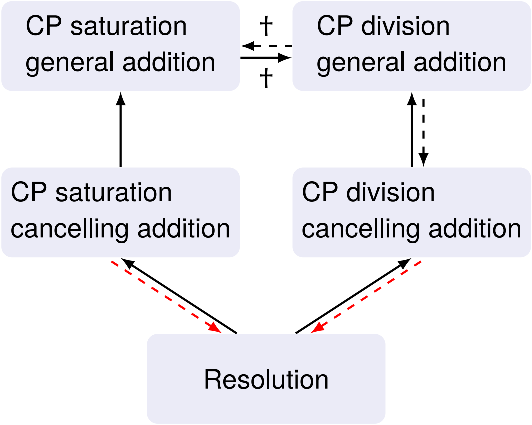

Pseudo-Boolean versions of

- Pigeonhole principle
- Subset cardinality
- …

have proof of size

- polynomial in all CP subsystems
- exponential in resolution

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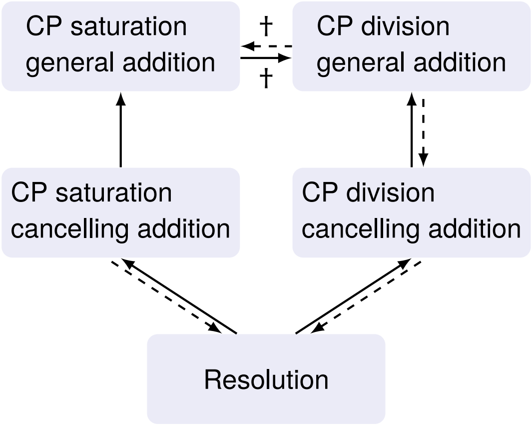
Proof Systems



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Pseudo-Boolean versions of

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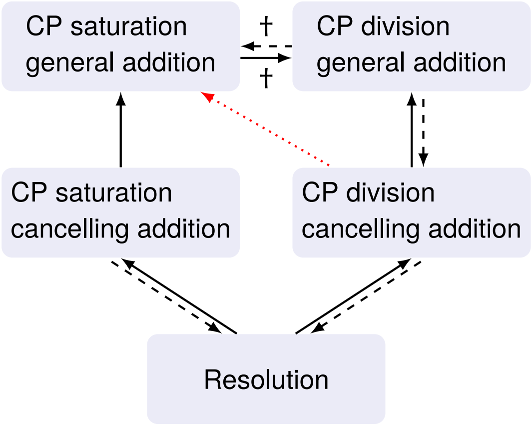
CNF version exponential $\Rightarrow$
Cannot recover encoding $\Rightarrow$
Subsystems are incomplete

Proof Systems



Separation candidates
Some formulas have proof of size

- polynomial in CP with cancelling addition and division
- unknown in CP with general addition and saturation

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)
$A \cdots\rightarrow B$: candidate for a separation
†: known only for polynomial-size coefficients

Take Home

Remarks

- Classified subsystems of Cutting Planes
- Saturation + Polynomial coefficients $\equiv$ Resolution
- Many formulas where PB solvers can shine

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Open problems

- Saturation $\equiv$ Resolution?
- Separation on PB inputs

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Thanks!