How Limited Interaction Hinders Real Communication (and What it Means for Proof and Circuit Complexity)

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joint work with Susanna F. de Rezende and Jakob Nordström

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Cutting Planes

Work with linear inequalities

$x \vee \overline{y} \quad \rightarrow \quad x + (1 - y) \geq 1 \quad \rightarrow \quad x - y \geq 0$

Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (a_i + b_i) x_i \geq a + b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a/k \rceil}$$

Goal: derive $0 \geq 1$

How to Measure Space

$$F = \{x + y \geq 1,\ y - x \geq 0,\ -y \geq 0\}$$

Present proof on board.

- Write down axiom inequalities
- Infer new inequalities
- Erase inequalities to save space

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$$y \geq 1$$
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Complexity Measures

Size # bits in proof

- Size $2^{O(N)}$ always possible.

Length # lines in proof

- Worst case $2^{\Omega(N^{\epsilon})}$. [Pudlák '97]

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Total space max # bits in memory at the same time

- Space $O(N^2)$ always possible; worst case $\Omega(N)$.

Line space max # lines in memory at the same time

- Space 5 always possible. [Galesi, Pudlák, Thapen '15]

Trade-offs

Question

Assume F has a proof in length L and another proof in space s.
Is there a proof in length $O(L)$ and space $O(s)$?

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Assume F has a proof in length L and another proof in space s.
Is there a proof in length $\mathrm{O}(L)$ and space $\mathrm{O}(s)$?

No

Trade-offs

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Assume F has a proof in length L and another proof in space s.
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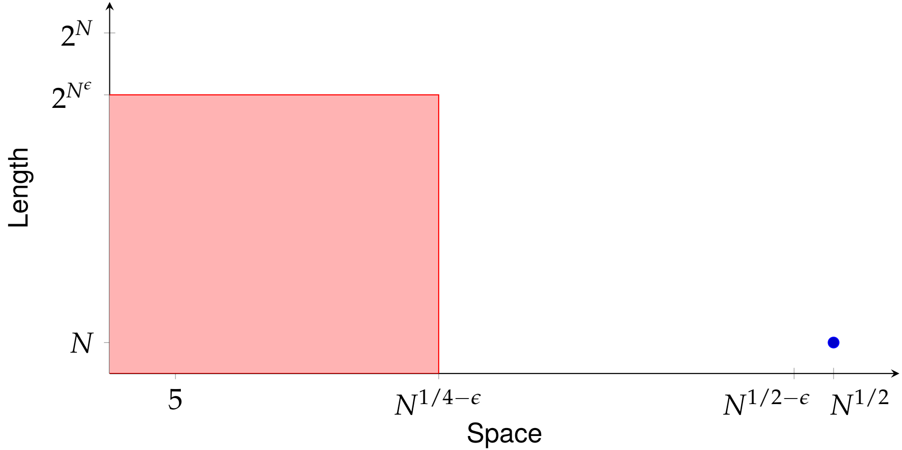
No

Previously studied for resolution and polynomial calculus

[Ben Sasson, Nordström '11] [Beame, Beck, Impagliazzo '12] [Beck, Nordström, Tang '13]

Trade-offs

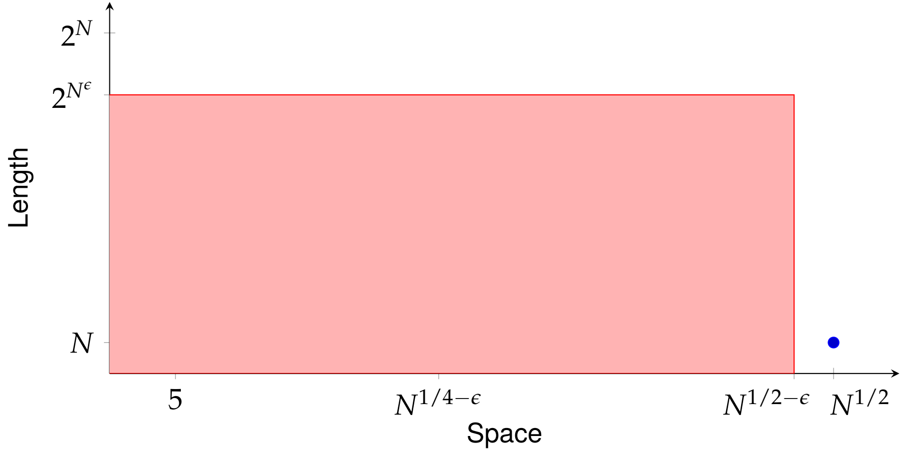


[Huynh, Nordström '12]

Can do length $\mathrm{O}(N)$, space $N^{1/2}$.

But space $N^{1/4-\epsilon}$ requires size $\exp(N^{\epsilon-\mathrm{o}(1)})$.

Trade-offs

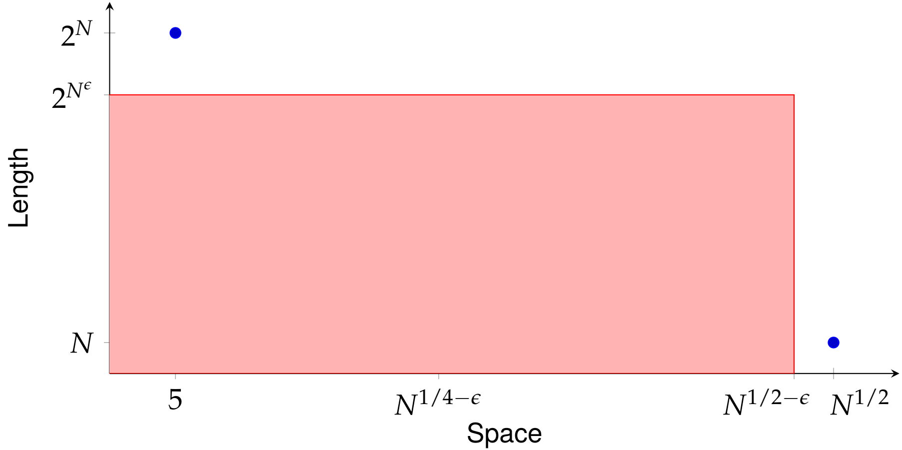


[Göös, Pitassi '14]

Can do length $N^{1+o(1)}$, space $N^{1/2+o(1)}$.

But space $N^{1/2-\epsilon}$ requires size $\exp(N^{\epsilon-o(1)})$.

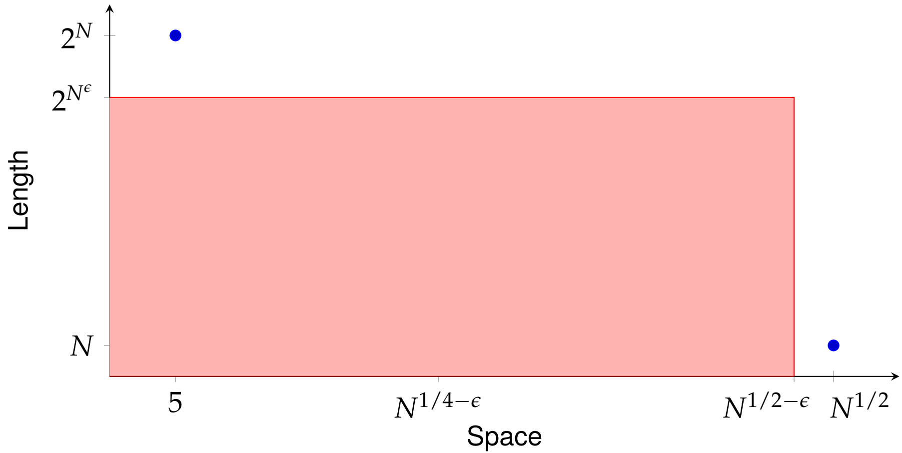
Trade-offs



[Galesi, Pudlák, Thapen ’15]

Can do length 2^N, space 5.

Trade-offs



[Galesi, Pudlák, Thapen '15]

Can do length 2^N, space 5.

But exponential coefficients and quadratic total space.

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Assume F has a proof in small total space with polynomial coefficients. Are there still trade-offs?

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This talk:

Yes

Trade-offs

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This talk:

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Cannot answer with previous techniques (provably)

Main Result

Theorem

There is a family of 6-CNF formulas with

- *short proofs: size* $\mathrm{O}(N)$, *total space* $\mathrm{O}(N^{2/5})$;

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- *but line space* $N^{1/20-\epsilon}$ *requires length* $\exp(\Omega(N^{1/40}))$.

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- Upper bounds with constant coefficients, counting all bits.
- Lower bound with unbounded coefficients, only counting lines.
- Lower bound for semantic cutting planes.

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Theorem

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- Upper bounds with constant coefficients, counting all bits.
- Lower bound with unbounded coefficients, only counting lines.
- Lower bound for semantic cutting planes.

- Holds for resolution and polynomial calculus proof systems.

Spin-off

Exponential separation of the monotone-AC hierarchy

Theorem

There is a monotone Boolean function with

- *small monotone circuits: size* $\mathrm{O}(n)$*, depth* $\log^{i}(n)$*, fan-in* $n^{4/5}$
- *but monotone circuits of depth* $\mathrm{O}(\log^{i-1} n)$ *require size* $\exp(\Omega(n^{\epsilon}))$*.*

Superpolynomial separation known [Raz, McKenzie '97]

Devious Plan

Assume refutation in length L and space s

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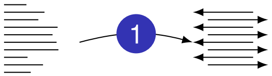
$\downarrow$

1. Communication protocol for falsified clause search problem



Devious Plan

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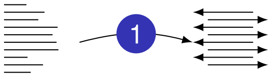
1. Communication protocol for Search(F)



Devious Plan

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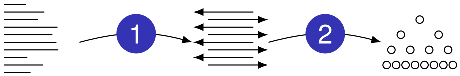

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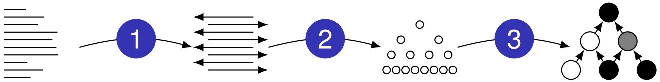
3. Strategy for Dymond–Tompa pebble game



Devious Plan

Assume refutation in length L and space s

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1. Communication protocol for $\text{Search}(F)$

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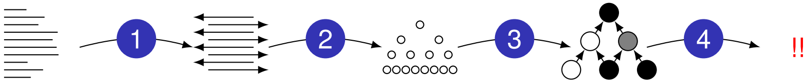
4. Construct graph with trade-offs



Devious Plan ❶: Proof $\rightarrow$ Protocol

Refutation in length L, space $s \rightarrow$
Protocol for Search(F) in $\log L$ rounds, communication $s \log L$

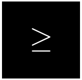
- Inspired by [Beame, Pitassi, Segerlind '05] [Beame, Huynh, Pitassi '10], explicit in [Huynh, Nordström '12].
- Key twists:
  - Real communication model
  - Measure number of rounds

Real Communication

Introduced by [Krajíček '98] to study cutting planes

- Compare real numbers at cost 1

Alice



Referee

Bob

Real Communication

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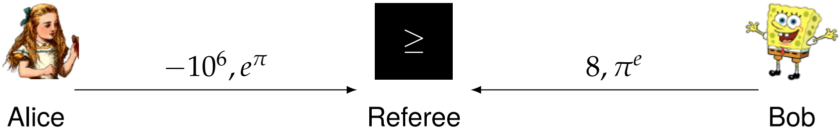

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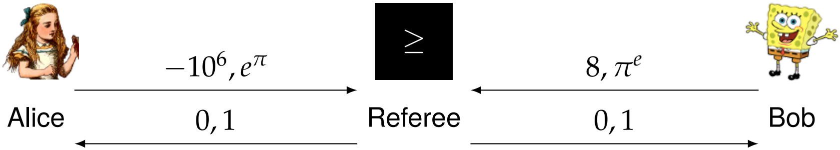

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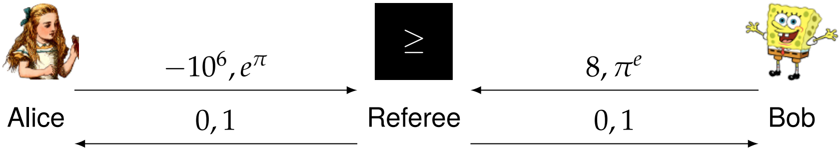

- Simulates deterministic communication (Alice sends m, Bob sends $1/2$)
- Stronger than deterministic communication (EQ)

Devious Plan ❶: Proof $\to$ Protocol

Falsified clause search on CNF $F(x, y)$

- Alice $\leftarrow$ assignment to x variables
- Bob $\leftarrow$ assignment to y variables
- Task: Find falsified clause

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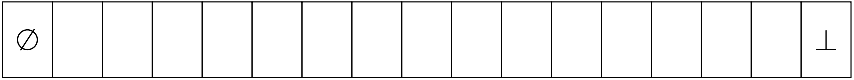

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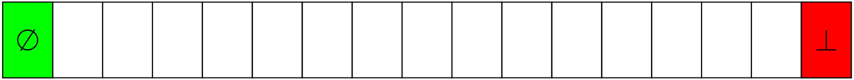

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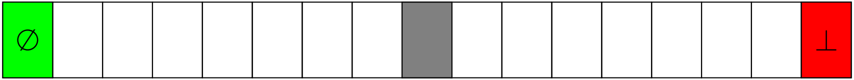

- Alice evaluates $\sum a_i x_i - a$ in s inequalities
- Bob evaluates $-\sum a_i y_i$ in s inequalities
- $\alpha(\mathbb{C}) = 1$ iff Referee answers $111\ldots1$

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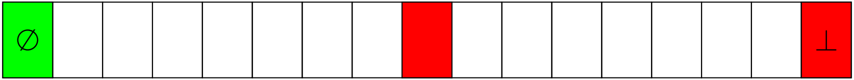

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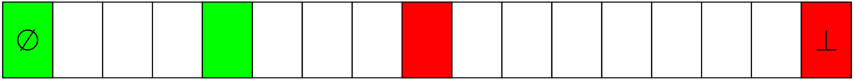

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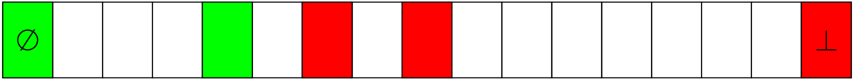

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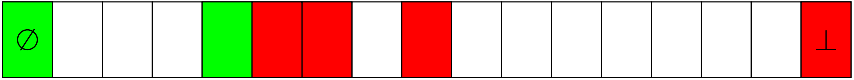

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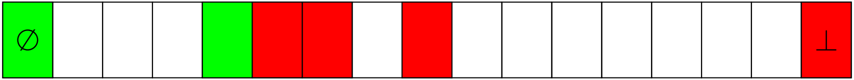

- $\alpha(\mathbb{C}) = 1 \quad \alpha(\mathbb{C} \cup \{A\}) = 0 \quad \Rightarrow \quad \alpha(A) = 0$
- $\log L$ rounds, communication $s \log L$

Devious Plan

Assume refutation in length L and space s

1. Communication protocol for Search(F) in $\log L$ rounds and communication $s \log L$
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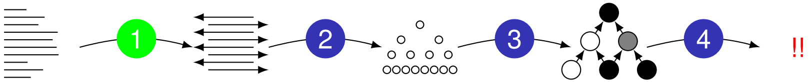


Devious Plan ❷: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
Parallel decision tree for S of depth r, c queries

Lifted Problem

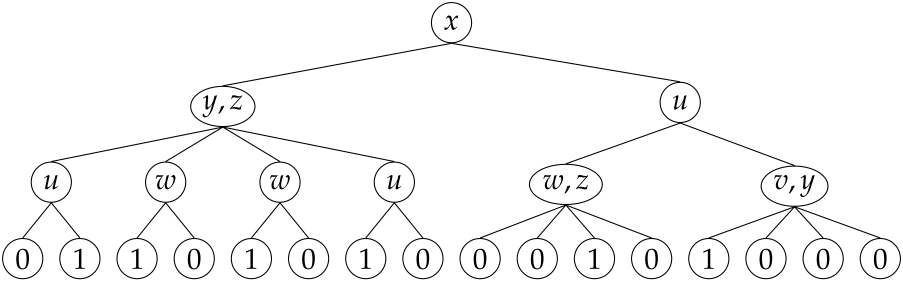
- Function $f(z_1, \ldots, z_n)$
- Alice $\leftarrow$ n indices $x_1, \ldots, x_n$
- Bob $\leftarrow$ n arrays $y_1, \ldots, y_n$

$z_1 = y_1[5] = 1$	5	0	0	1	0	0	1	1	1
$z_2 = y_2[1] = 0$	1	1	0	0	1	0	1	1	1

- Lifted function $\text{Lift}(f)(x, y) = f(y_1[x_1], \ldots, y_n[x_n])$

Parallel Decision Trees

Decision tree with many queries per node [Valiant ’75]



Depth Longest branch

Queries # queries in a branch

Devious Plan ❷: Protocol $\to$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\to$
Parallel decision tree for S of depth r, c queries

Devious Plan 2: Protocol $\leftarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\leftarrow$
Parallel decision tree for S of depth r, c queries

Communication

Decision tree
Query $\{z_3, z_{28}\}$

Devious Plan 2: Protocol $\leftarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\leftarrow$
Parallel decision tree for S of depth r, c queries

Communication

Alice sends x_3, x_{28}
Bob sends $y_3[x_3], y_{28}[x_{28}]$

Decision tree

Query $\{z_3, z_{28}\}$

Devious Plan 2: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
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Communication

Alice sends $x_1 + x_2 + \cdots + x_n$

Decision tree

Devious Plan ❷: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
Parallel decision tree for S of depth r, c queries

Communication

Alice sends $x_1 + x_2 + \cdots + x_n$

Decision tree

???

Devious Plan ❷: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
Parallel decision tree for S of depth r, c queries

- Main technical result (Simulation Theorem)
  - Technique from [Raz, McKenzie '97]
  - Adapted to real communication in [Bonet, Esteban, Galesi, Johannsen '98]
  - Connection to decision trees made explicit in [Göös, Pitassi, Watson '15]
- Our contribution
  - Introduce rounds
  - Adapt to real communication preserving rounds

Devious Plan

Assume refutation of lifted formula in length L and space s

1. Communication protocol for Lift$(\text{Search}(F))$ in $\log L$ rounds and communication $s \log L$
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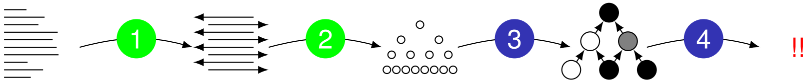
3. Strategy for Dymond–Tompa pebble game

4. Construct graph with trade-offs



Devious Plan

Assume refutation of lifted pebbling formula in length L and space s

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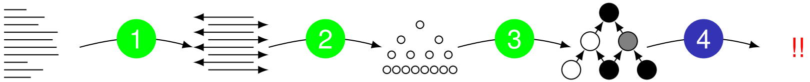
3. Strategy for Dymond–Tompa pebble game for $\log L$ rounds and $s \log L$ pebbles [Chan '13]
4. Construct graph with trade-offs



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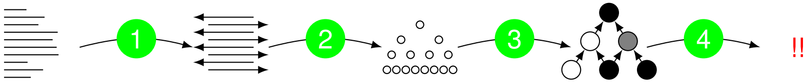
4. Construct graph where such strategy does not exist



Take Home

Remarks

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- Hold for resolution, polynomial calculus, cutting planes
- Key to measure rounds

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Open problems

- Smaller lift size
- Stronger models of communication

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Thanks!