How Limited Interaction Hinders Real Communication (and What it Means for Proof and Circuit Complexity)

Marc Vinyals

KTH Royal Institute of Technology
Stockholm, Sweden

joint work with Susanna F. de Rezende and Jakob Nordström

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When are SAT Solvers Not Good?

Proof complexity

- Examples of hard formulas. . . yes, we knew that
- Examples of easy formulas. . . but hard in practice!

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- Proofs may not be easy to find
- Simulation results do not allow forgetting clauses
- This talk: aggressive memory minimization is dangerous

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Disclaimer

- Theoretical work, no experiments

Trade-offs

Question

Assume F has a proof in length L and another proof in space s.
Is there a proof in length $\mathrm{O}(L)$ and space $\mathrm{O}(s)$?

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No

- Studied for resolution, polynomial calculus, and a model of CDCL
  [Ben Sasson, Nordström '11] [Beame, Beck, Impagliazzo '12] [Beck, Nordström, Tang '13] [Elffers, Johannsen, Lauria, Magnard, Nordström, V '16]
- This talk: cutting planes

Cutting Planes

Work with inequalities

$$x \vee \overline{y} \quad \to \quad x + (1 - y) \geq 1 \quad \to \quad x - y \geq 0$$

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Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (a_i + b_i) x_i \geq a + b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a/k \rceil}$$

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Goal: derive $0 \geq 1$

Complexity Measures

Size # bits in proof

- Size $2^{O(N)}$ always possible.

Length # lines in proof

- Worst case $2^{\Omega(N^\epsilon)}$. [Pudlák '97]

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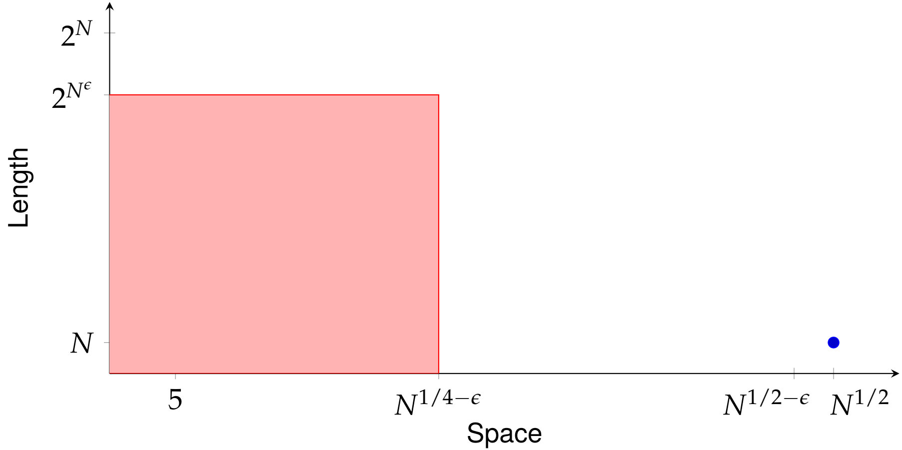
Total space max # bits in memory at the same time

- Space $O(N^2)$ always possible; worst case $\Omega(N)$.

Line space max # lines in memory at the same time

- Space 5 always possible. [Galesi, Pudlák, Thapen '15]

Trade-offs

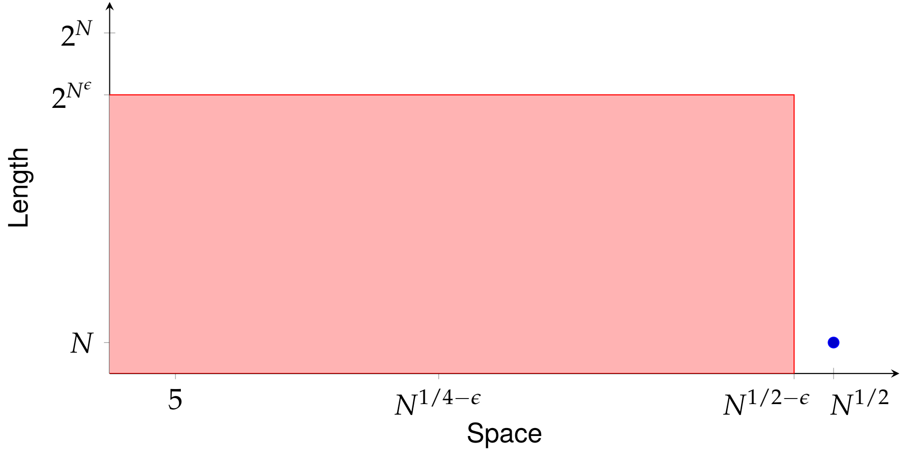


[Huynh, Nordström ’12]

Can do length $\mathrm{O}(N)$, space $N^{1/2}$.

But space $N^{1/4-\epsilon}$ requires size $\exp(N^{\epsilon-\mathrm{o}(1)})$.

Trade-offs

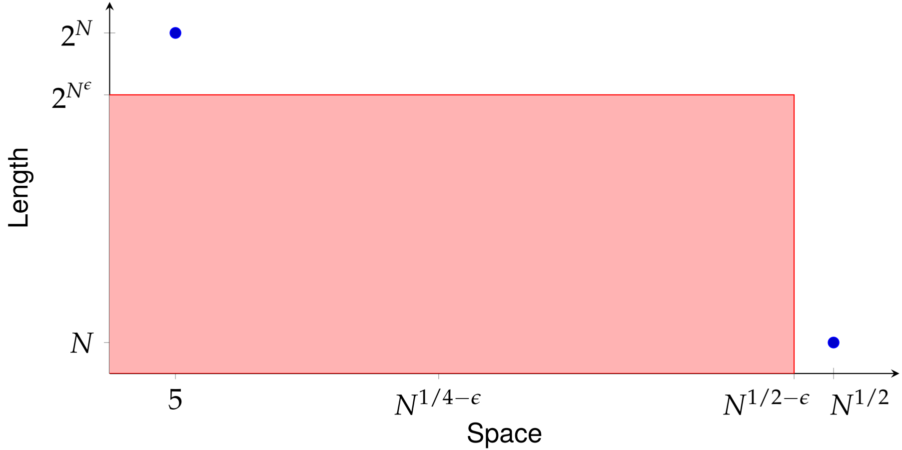


[Göös, Pitassi '14]

Can do length $N^{1+o(1)}$, space $N^{1/2+o(1)}$.

But space $N^{1/2-\epsilon}$ requires size $\exp(N^{\epsilon-o(1)})$.

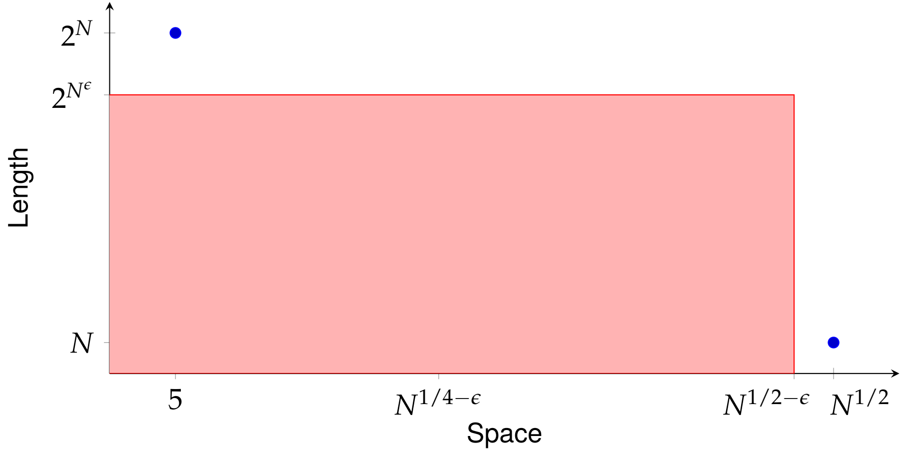
Trade-offs



[Galesi, Pudlák, Thapen '15]

Can do length 2^N, space 5.

Trade-offs



[Galesi, Pudlák, Thapen '15]

Can do length 2^N, space 5.

But exponential coefficients and quadratic total space.

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Assume F has a proof in small total space with polynomial coefficients.
Are there still trade-offs?

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This talk:

Yes

Main Result

Theorem

There is a family of 6-CNF formulas with

- *short proofs: size* $\mathrm{O}(N)$, *total space* $\mathrm{O}(N^{2/5})$;

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There is a family of 6-CNF formulas with

- *short proofs: size $O(N)$, total space $O(N^{2/5})$;*
- *small space proofs: total space $O(N^{1/40})$, size $2^{O(N^{1/40})}$;*
- *but line space $N^{1/20-\epsilon}$ requires length $\exp(\Omega(N^{1/40}))$.*

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- Upper bounds with constant coefficients, counting all bits.
- Lower bound with unbounded coefficients, only counting lines.
- Lower bound for semantic cutting planes.

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- Upper bounds with constant coefficients, counting all bits.
- Lower bound with unbounded coefficients, only counting lines.
- Lower bound for semantic cutting planes.
- Holds for resolution and polynomial calculus proof systems.

Spin-off

Exponential separation of the monotone-AC hierarchy

Theorem

There is a monotone Boolean function with

- *small monotone circuits: size* $O(n)$, *depth* $\log^i(n)$, *fan-in* $n^{4/5}$
- *but monotone circuits of depth* $O(\log^{i-1} n)$ *require size* $\exp(\Omega(n^\epsilon))$.

Superpolynomial separation known [Raz, McKenzie '97]

Devious Plan

Assume refutation in length L and space s

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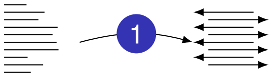
$\downarrow$

1. Communication protocol for falsified clause search problem



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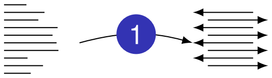
1. Communication protocol for Search(F)



Devious Plan

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1. Communication protocol for Search(F)

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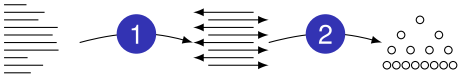

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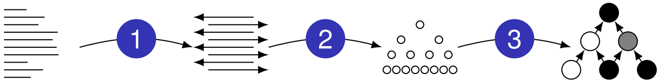

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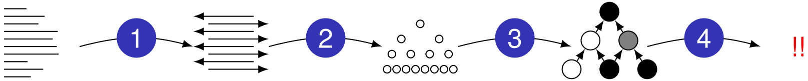
4. Construct graph with trade-offs



Devious Plan ➊: Proof $\rightarrow$ Protocol

Refutation in length L, space s $\rightarrow$
Protocol for Search(F) in $\log L$ rounds, communication $s \log L$

- Inspired by [Beame, Pitassi, Segerlind '05] [Beame, Huynh, Pitassi '10], explicit in [Huynh, Nordström '12].
- Key twists:
  - Real communication model
  - Measure number of rounds

Communication Complexity

$f(x,y)$?

x

y

- Two parties compute $f(x,y)$
- Alice knows $x \in X$, Bob knows $y \in Y$

Communication Complexity

x $\longrightarrow m_1(x) \longrightarrow$ y

$\longleftarrow m_2(y, m_1) \longleftarrow$

$\longrightarrow m_3(x, m_1, m_2) \longrightarrow$

$\vdots$

$\longrightarrow m_k(x, m_1, \ldots, m_{k-1}) = f(x, y) \longrightarrow$

- Two parties compute $f(x, y)$
- Alice knows $x \in X$, Bob knows $y \in Y$
- Communicate alternately

Communication Complexity

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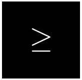
- Two parties compute $f(x, y)$
- Alice knows $x \in X$, Bob knows $y \in Y$
- Communicate alternately
- Cost = # bits sent in worst case
- Rounds = # messages sent in worst case

Real Communication

Introduced in [Krajíček ’98] to study cutting planes

- Compare real numbers at cost 1

Alice



Referee

Bob

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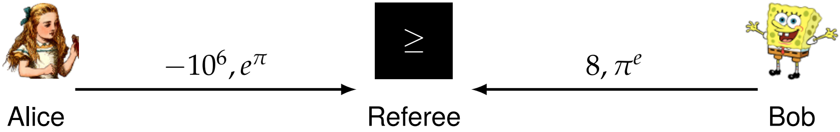

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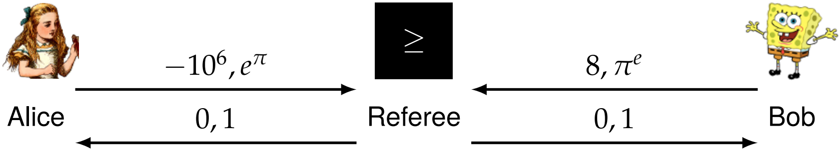

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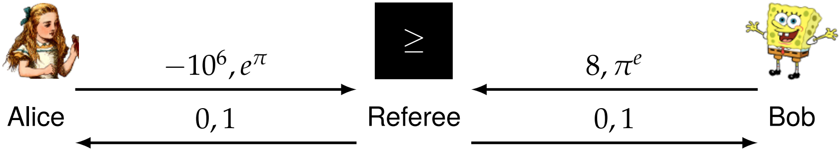

- Simulates deterministic communication (Alice sends m, Bob sends $1/2$)
- Stronger than deterministic communication (EQ)

Devious Plan 1: Proof $\rightarrow$ Protocol

Falsified clause search on CNF $F(x, y)$

- Alice $\leftarrow$ assignment to x variables
- Bob $\leftarrow$ assignment to y variables
- Task: Find falsified clause

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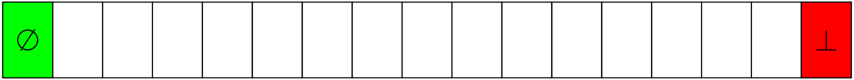

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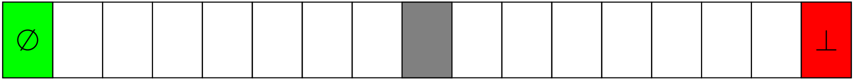

- Alice evaluates $\sum a_i x_i - a$ in s inequalities
- Bob evaluates $-\sum a_i y_i$ in s inequalities
- $\alpha(\mathbb{C}) = 1$ iff Referee answers $111\ldots1$

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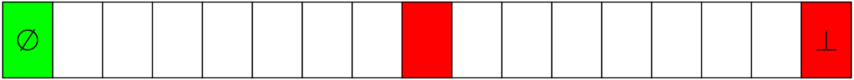

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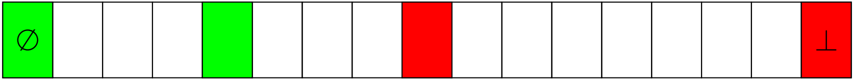

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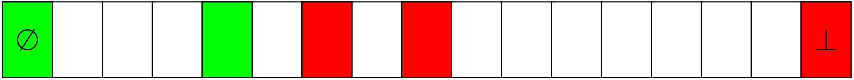

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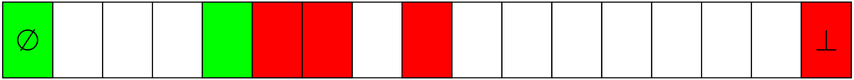

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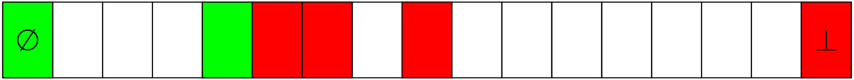
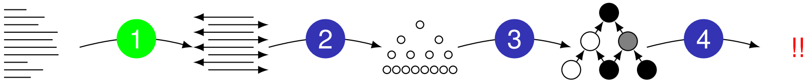


- $\alpha(\mathbb{C}) = 1 \quad \alpha(\mathbb{C} \cup \{A\}) = 0 \quad \Rightarrow \quad \alpha(A) = 0$
- $\log L$ rounds, communication $s \log L$

Devious Plan

Assume refutation in length L and space s

1. Communication protocol for Search(F) in $\log L$ rounds and communication $s \log L$
2. Parallel decision tree for Search(F)
3. Strategy for Dymond–Tompa pebble game
4. Construct graph with trade-offs



Devious Plan 2: Protocol $\rightarrow$ Decision Tree

Protocol for Lift(S) in r rounds, communication c $\rightarrow$
Parallel decision tree for S of depth r, c queries

- Main technical result (Simulation Theorem)
  - Technique from [Raz, McKenzie '97]
  - Adapted to real communication in [Bonet, Esteban, Galesi, Johannsen '98]
  - Connection to decision trees made explicit in [Göös, Pitassi, Watson '15]
- Our contribution
  - Introduce rounds
  - Adapt to real communication preserving rounds

Devious Plan

Assume refutation of lifted formula in length L and space s

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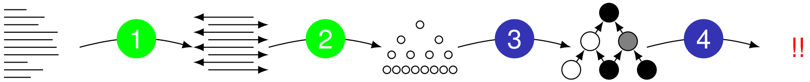

Devious Plan

Assume refutation of lifted pebbling formula in length L and space s

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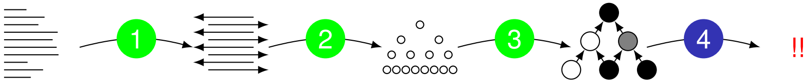
3. Strategy for Dymond–Tompa pebble game for $\log L$ rounds and $s \log L$ pebbles [Chan '13]
4. Construct graph with trade-offs



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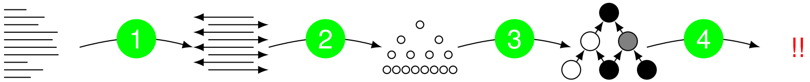
4. Construct graph where such strategy does not exist



Take Home

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- Key to measure rounds

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- Verify experimentally
- Smaller lift size
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Thanks!