Cumulative Space in Black-White Pebbling and Resolution

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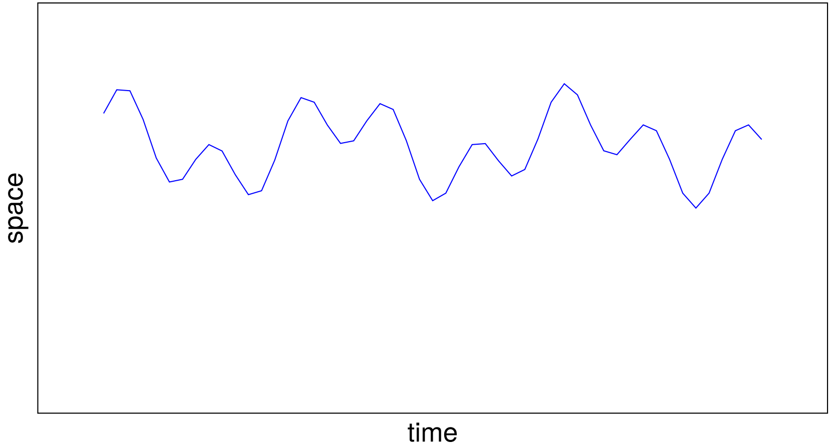
8th Innovations in Theoretical Computer Science

What is space?



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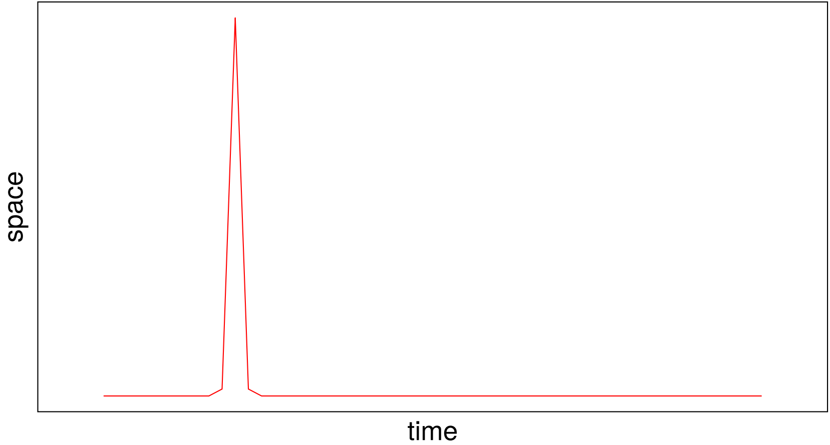

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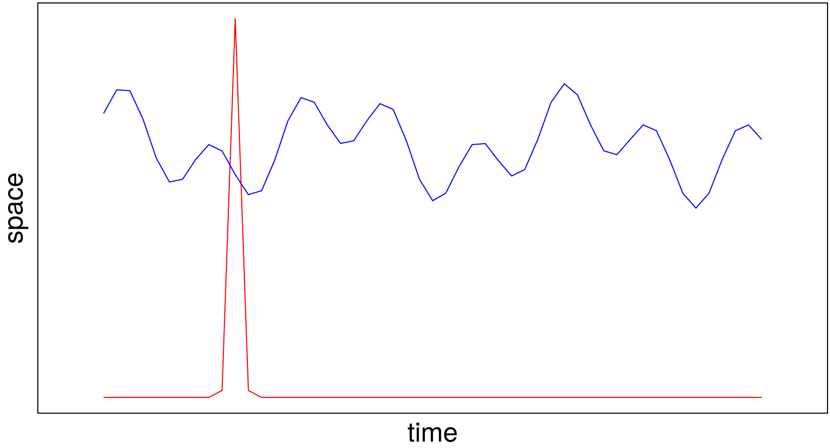

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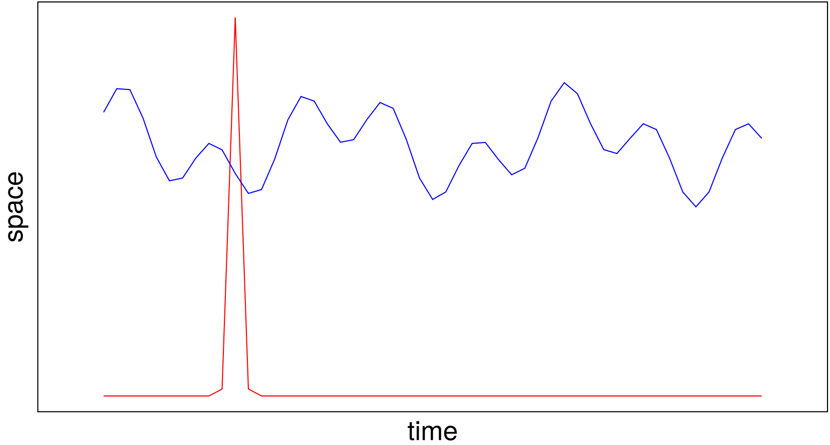

Usually: *maximal* space.

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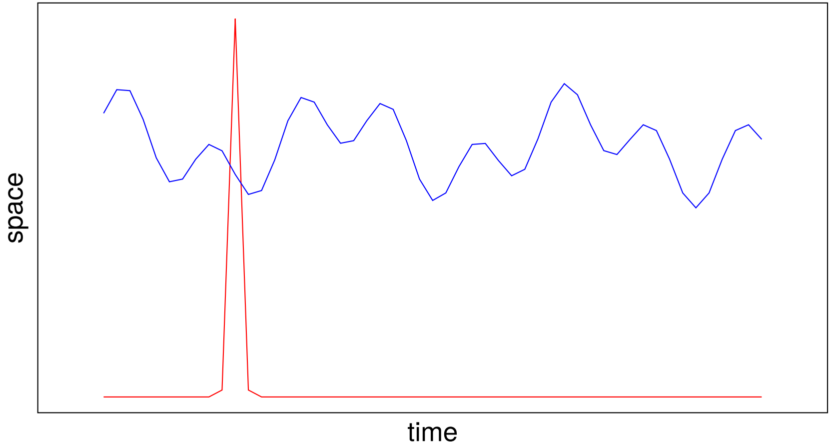

Usually: *maximal* space.

[Alwen, Serbinenko '15]: aggregate space over computation (*cumulative* space).

Resolution

Setup

Prove CNF formula unsatisfiable.

Present proof on board.

- Write down axiom clauses
- Infer new clauses
  $$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$
- Erase clauses to save space

Goal: derive empty clause $\perp$

$$F = \{x,\ \overline{x} \vee y,\ \overline{y}\}$$

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$$\cancel{x}$$
$$\overline{x} \vee y$$
$$y$$

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y

$\overline{y}$

$\perp$

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$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$
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$$\boxed{\begin{array}{c} y \\ \overline{y} \\ \perp \end{array}}$$

Goal: derive empty clause $\perp$

Questions

- How much time will this take? (Length)
- How large is the blackboard? (Space)

Space

[Esteban, Torán '99]

[Alekhnovich, Ben Sasson, Razborov, Wigderson '00]

x	x $\overline{x} \vee y$	x $\overline{x} \vee y$ y	$\cancel{x}$ $\overline{x} \vee y$ y	$\cancel{\overline{x} \vee y}$ y	y $\overline{y}$	y $\overline{y}$ $\perp$

$$|\mathbb{C}_1| = 1 \ |\mathbb{C}_2| = 2 \ |\mathbb{C}_3| = 3 \ |\mathbb{C}_4| = 2 \ |\mathbb{C}_5| = 1 \ |\mathbb{C}_6| = 2 \ |\mathbb{C}_7| = 3$$

Space of a proof: $\mathrm{Sp}(\pi) := \max_t |\text{Clauses in } \mathbb{C}_t| = 3$

Space of refuting a formula: $\mathrm{Sp}(F \vdash \perp) := \min_{\pi :\, F \vdash \perp} \mathrm{Sp}(\pi) \leq 3$

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x	x $\overline{x} \vee y$	x $\overline{x} \vee y$ y	$\cancel{x}$ $\overline{x} \vee y$ y	$\cancel{\overline{x} \vee y}$ y	y $\overline{y}$	y $\overline{y}$ $\perp$
$\lvert\mathbb{C}_1\rvert = 1$	$\lvert\mathbb{C}_2\rvert = 2$	$\lvert\mathbb{C}_3\rvert = 3$	$\lvert\mathbb{C}_4\rvert = 2$	$\lvert\mathbb{C}_5\rvert = 1$	$\lvert\mathbb{C}_6\rvert = 2$	$\lvert\mathbb{C}_7\rvert = 3$

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Alternative measures: # literals, # bits

Space

Bounds

Every formula $\mathrm{Sp} = \mathrm{O}(n)$

Exist formulas st $\mathrm{Sp} = \Omega(n)$

[Esteban, Torán '99], [Alekhnovich, Ben Sasson, Razborov, Wigderson '00]

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Space vs length

Exist formulas st

- Exists proof with $\mathrm{Sp} = \mathrm{O}(n^{1/11})$
- Exists proof with $\mathrm{Len} = \mathrm{O}(n)$
- Every proof with $\mathrm{Sp} < n^{2/11}$ requires $\mathrm{Len} = \exp n^{\Omega(1)}$

[Ben Sasson, Nordström '11]

Cumulative Space

Aggregate space over whole proof.

x	x $\overline{x} \lor y$	x $\overline{x} \lor y$ y	$\cancel{x}$ $\overline{x} \lor y$ y	$\cancel{\overline{x} \lor y}$ y	y $\overline{y}$	y $\overline{y}$ $\perp$
$\lvert\mathbb{C}_1\rvert = 1$	$\lvert\mathbb{C}_2\rvert = 2$	$\lvert\mathbb{C}_3\rvert = 3$	$\lvert\mathbb{C}_4\rvert = 2$	$\lvert\mathbb{C}_5\rvert = 1$	$\lvert\mathbb{C}_6\rvert = 2$	$\lvert\mathbb{C}_7\rvert = 3$

Cumulative space of a proof: $\mathrm{CumSp}(\pi) := \sum_t |\text{Clauses in } \mathbb{C}_t| = 14$

Cumulative space of refuting a formula:

$\mathrm{CumSp}(F \vdash \perp) := \min_{\pi:\, F \vdash \perp} \mathrm{CumSp}(\pi) \leq 14$

Cumulative Space

Observations

Every proof $\mathrm{CumSp} \leq \mathrm{Len} \cdot \mathrm{Sp}$
Every formula $\mathrm{Len} \leq 2^n$ and $\mathrm{Sp} \leq n$
$\Rightarrow \mathrm{CumSp} \leq n2^n$

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$\Rightarrow$ Most interesting if $\mathrm{Len} = \mathrm{O}(n)$.

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Every formula $\mathrm{CumSp} \leq \mathrm{Len}^2$.

Reaching space s needs $s/2$ configurations of space $\geq s/2$
$\Rightarrow$ Cumulative space $\Omega(s^2)$.

Cumulative Space Bounds

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Maximal space: $\mathrm{Sp} = \mathrm{O}(\mathrm{Len})$ not tight.
Every formula $\mathrm{Sp} = \mathrm{O}(\mathrm{Len}/\log\mathrm{Len})$. [Hopcroft, Paul, Valiant '75]

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Every formula $\mathrm{Sp} = \mathrm{O}(\mathrm{Len} / \log \mathrm{Len})$. [Hopcroft, Paul, Valiant '75]

Theorem

Exist formulas with $\mathrm{Len} = \mathrm{O}(n)$ and $\mathrm{CumSp} = \Omega(n^2)$.

Maximal vs Cumulative Space

Large space $\Leftrightarrow$ large cumulative space?

$\Rightarrow$ Yes

Every formula $\mathrm{CumSp} = \Omega(\mathrm{Sp}^2)$.

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Every formula $\mathrm{CumSp} = \Omega(\mathrm{Sp}^2)$.

$\Leftarrow$ No

Theorem

Exist formulas with $\mathrm{Sp} = \mathrm{O}(\log n)$ but $\mathrm{CumSp} = \Omega(n^2 / \log n)$

Length vs Cumulative Space

How often do we need maximum space in a trade-off?

Theorem [Ben Sasson, Nordström '11]

Exist formulas st for any $s = \mathrm{O}(\sqrt{n})$

- Exists proof with $\mathrm{Sp} = \mathrm{O}(s)$ and $\mathrm{Len} = \mathrm{O}(n^2/s^2)$
  - Exists proof with $\mathrm{Sp} = \mathrm{O}(1)$
  - Exists proof with $\mathrm{Len} = \mathrm{O}(n)$
- Every proof in space $\mathrm{O}(s)$ needs $\mathrm{Len} = \Omega(n^2/s^2)$

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- Every proof in space $\mathrm{O}(s)$ needs $\mathrm{Len} \cdot \mathrm{Sp} = \Omega(n^2/s)$

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How often do we need maximum space in a trade-off?

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Exist formulas st for any $s = \mathrm{O}(\sqrt{n})$

- Exists proof with $\mathrm{Sp} = \mathrm{O}(s)$ and $\mathrm{Len} = \mathrm{O}(n^2/s^2)$
  - Exists proof with $\mathrm{Sp} = \mathrm{O}(1)$
  - Exists proof with $\mathrm{Len} = \mathrm{O}(n)$
- Every proof in space $\mathrm{O}(s)$ needs $\mathrm{CumSp} = \Omega(n^2/s)$

Corollary

- Every proof in space $\mathrm{O}(s)$ and length $\mathrm{O}(n^2/s^2)$ needs $\Omega(n^2/s^2)$ configurations with space $\Omega(s)$

Parallel Resolution

Parallel resolution: allow many steps at once.

Automatic $\mathrm{CumSp} = \Omega(\mathrm{Sp}^2)$ lower bound no longer holds.

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Fully Parallel Resolution

Very powerful model: can prove any formula in 2 steps.

Lower bounds with limited space.

Techniques

Pebble games

- Simple computational model to measure space.
- Prove lower bounds in pebble game
- Translate to resolution

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Lemma

Resolution proof of $F(G)$ in length L, space s, cumulative space c.
Then pebbling of G in time L, space s, cumulative space c.

Even if parallel inference steps.

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Pebble games

- Simple computational model to measure space.
- Prove lower bounds in pebble game
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Lemma

Resolution proof of $F(G)$ in length L, space s, cumulative space c.
Then pebbling of G in time L, space s, cumulative space c.

Even if parallel inference steps.

- [Alwen, Serbinenko '15]: Translate computation to black pebbling strategy.
- Proofs are non-deterministic: translate proof to black-white pebbling.

Take Home

Recap

- Introduced cumulative space measure in proof complexity.

Open problems

- Study cumulative space in other areas.

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Thanks!