CDCL vs Resolution

Marc Vinyals

DPLL

$y \vee z \quad y \vee \bar{z} \quad x \vee \bar{y} \vee z \quad x \vee \bar{y} \vee \bar{z} \quad \bar{x} \vee \bar{y}$

Algorithm 1: DPLL

```
while not solved do
    if conflict then backtrack()
    else if unit then propagate()
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```

State: partial assignment

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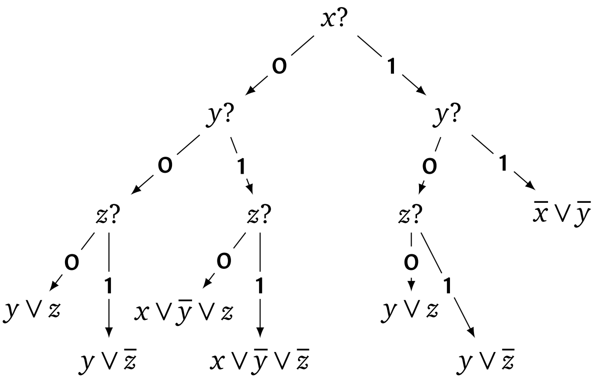

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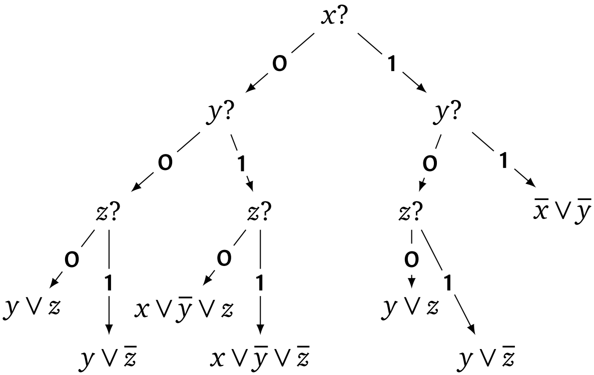
Resolution

- Interpret DPLL run as resolution proof

$$\frac{C \lor v \qquad D \lor \overline{v}}{C \lor D}$$



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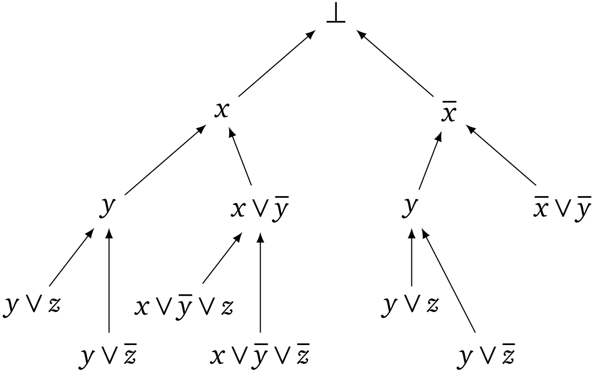

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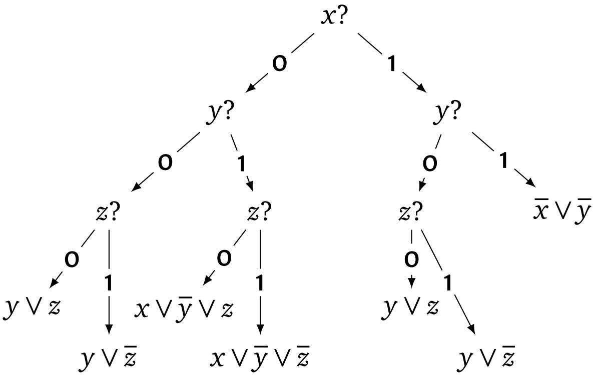

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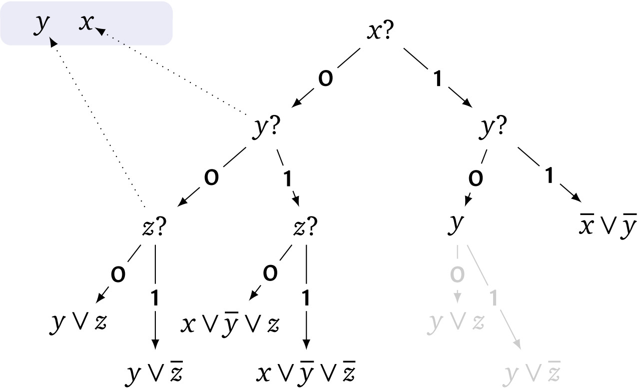
Algorithm 2: CDCL

while *not solved* **do**
- **if** *conflict* **then** learn()
- **else if** *unit* **then** propagate()
- **else**
  - maybe forget()
  - maybe restart()
  - branch()

State: partial assignment & learned clauses



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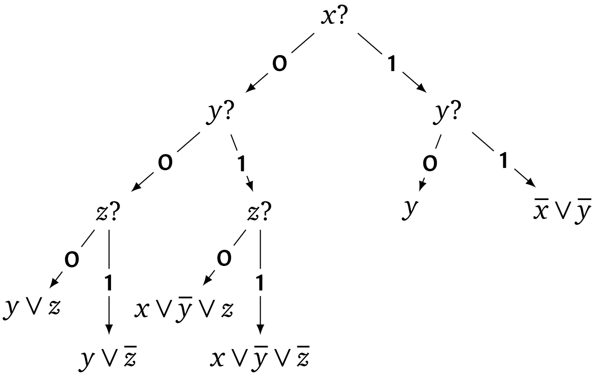

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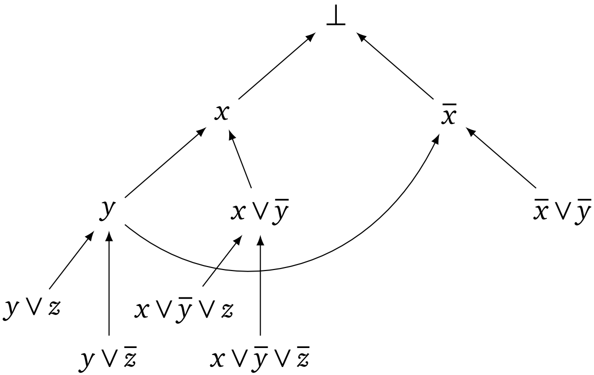

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- Yes (under natural model)
  [Pipatsrisawat, Darwiche '09]
  [Atserias, Fichte, Thurley '09]

CDCL equivalent to Resolution: Results

Theorem [Pipatsrisawat, Darwiche '09]

With **non-deterministic** variable decisions,
CDCL can efficiently find resolution proofs

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With **random** variable decisions,
CDCL can efficiently find **bounded-width** resolution proofs

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CDCL equivalent to Resolution: Simulation

- Derivation $\pi = C_1, \ldots, C_t$.
- Goal: learn every clause $C_i \in \pi$.

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$x \vee y$ not absorbed:

- if $x = 0$ then would propagate y, but DB does not.

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$x \lor y$ is absorbed:

- if $x = 0$ then propagate $z = 1$ and $y = 1$;
- if $y = 0$ then propagate $z = 1$ and $x = 1$.

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Algorithm 3: Simulation

for $C_i \in \pi$ **do**
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- Frequent restarts
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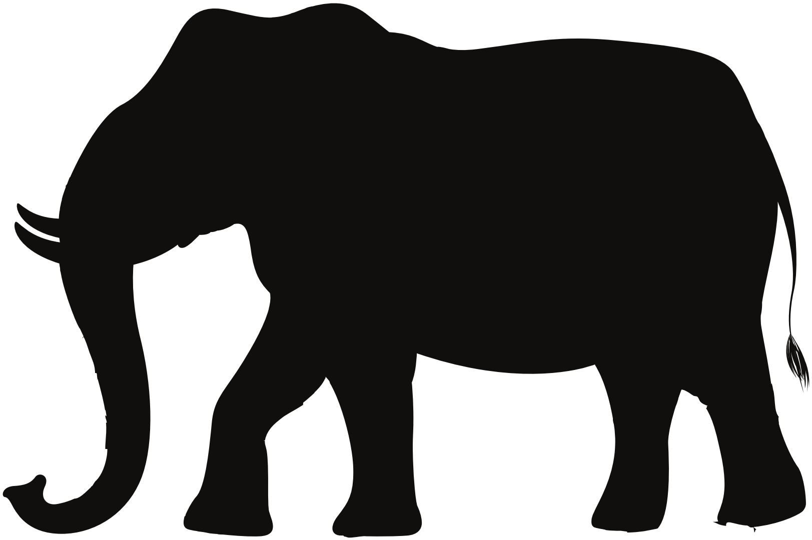

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Optimal variable choices are needed

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- CDCL with any static order exponentially worse than resolution.
  [Mull, Pang, Razborov '19]
- CDCL with VSIDS and similar heuristics exponentially worse than resolution.
  [V '20]

Hard Formulas for VSIDS

- Give a score $q = q(x)$ to variable x.
- At each conflict
  - Bump $q' = q + 1$ if x involved.
  - Decay $q' = 0.95 \cdot q$ all variables.
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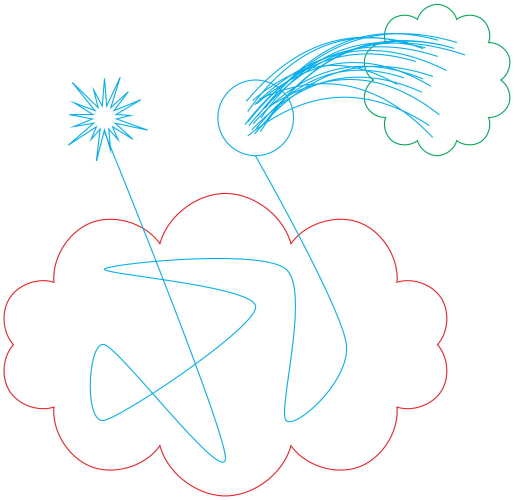

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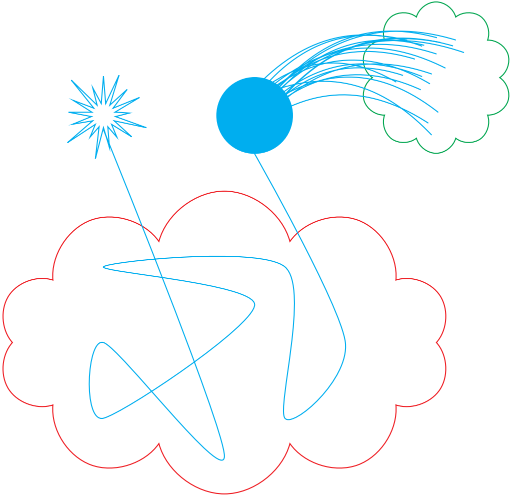

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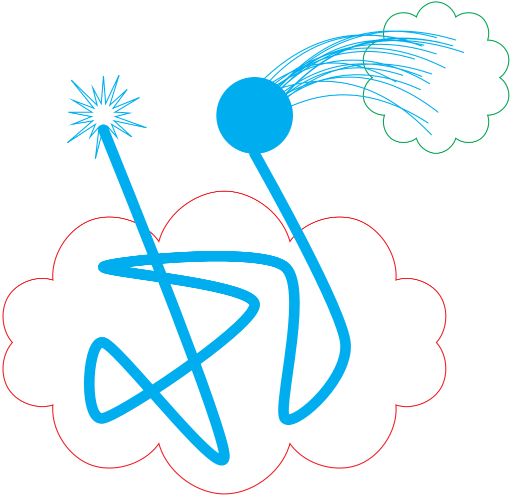

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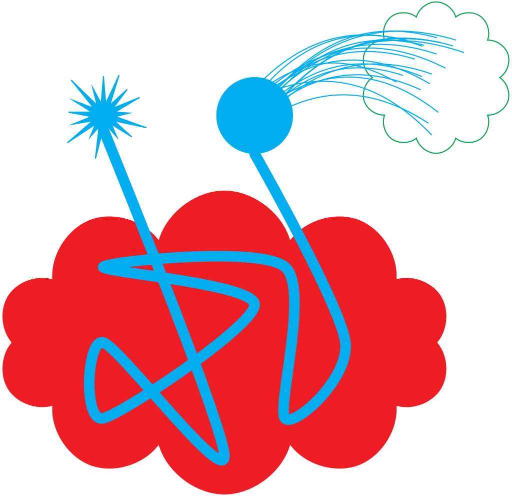

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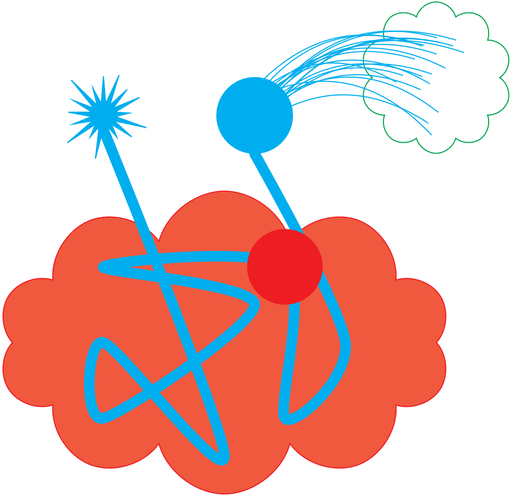

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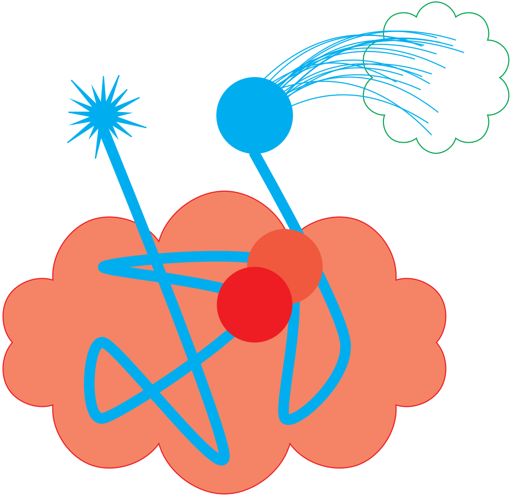

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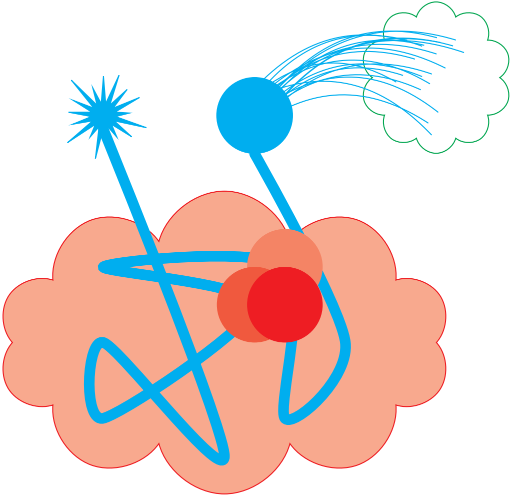

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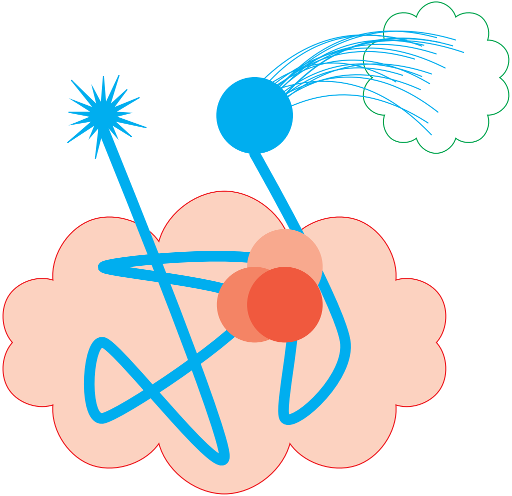

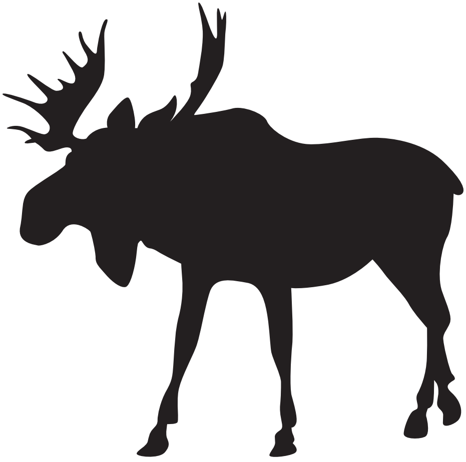

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Throwing Clauses Away

No great theoretical framework

- With nondeterministic erasures enough to keep only $n \ll L$ clauses in memory.
  [Esteban, Torán '01]
- But more are needed to simulate resolution:
- Keeping $\ll n$ clauses can exponentially blow-up runtime.
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- Keeping $\ll n^k$ clauses can superpolynomially blow-up runtime.
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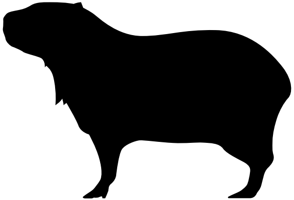
- Keeping only narrow clauses can exponentially blow-up runtime.

  [Thapen '16]

- What about clauses with low LBD?



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- CDCL without restarts between *regular* and *standard* resolution.

CDCL and Regular Resolution

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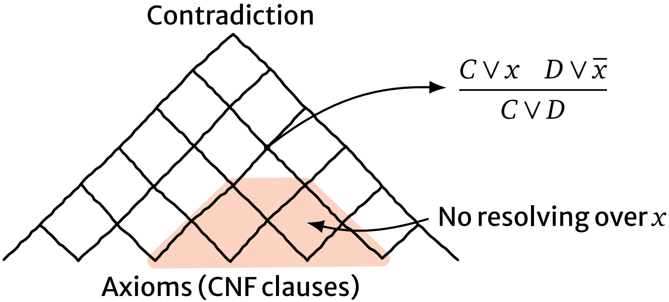

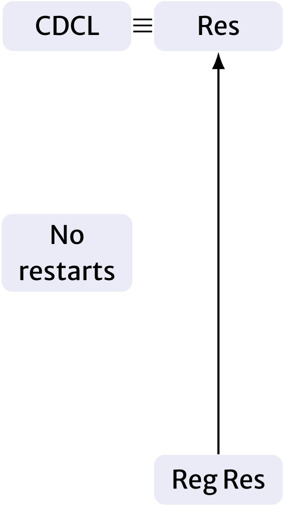

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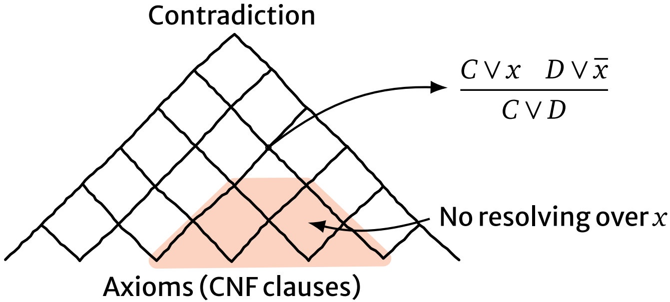

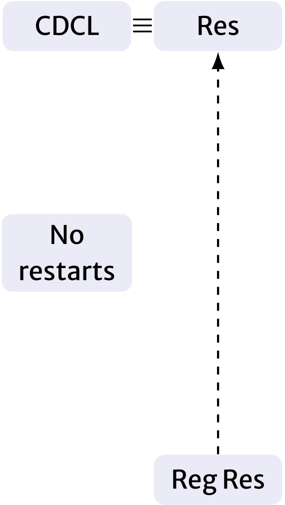

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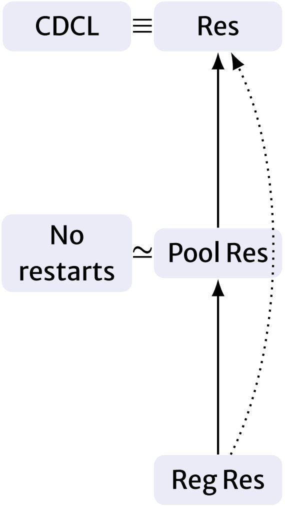

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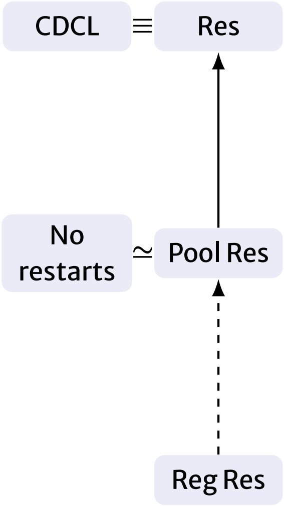

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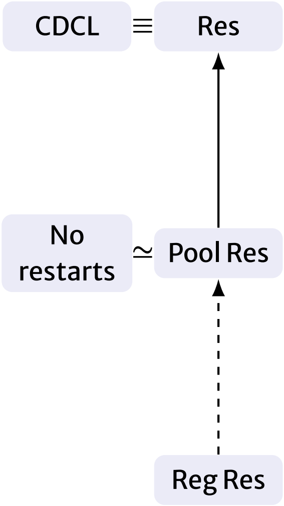
- Formula with CDCL proof of length L but requires $L + 1$ w/o restarts?



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- Less overhead with decision learning scheme.
- Is decision faster than 1UIP?
- What overhead is needed?

Merge Resolution

- A resolution step is a merge if C and D share a literal.

Merge

$$\frac{x \vee y \vee z \qquad x \vee y \vee \overline{z}}{x \vee y}$$

Not a merge

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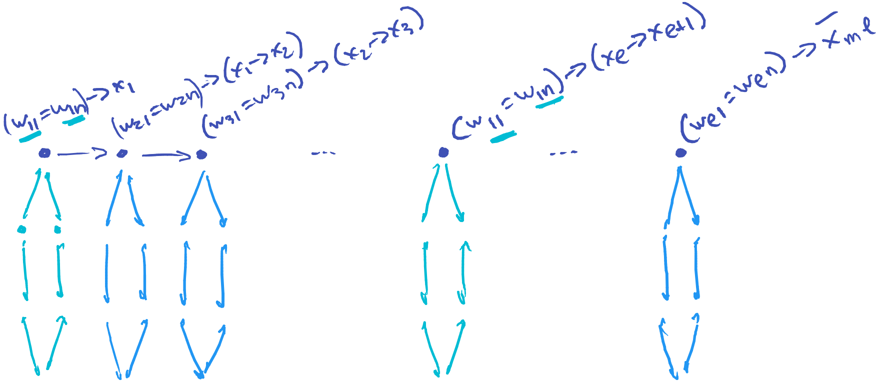
- Merge resolution 2.0: only reuse merges.
- 1UIP produces merge resolution proofs.
- Merge resolution can simulate standard resolution with $O(n)$ overhead.
- And $\Omega(n)$ overhead sometimes needed. [Fleming, Ganesh, Kolokolova, Li, V]

Tricky Formulas for Merge Resolution

$$\mathcal{W}: \quad w_j^k = w_j^{k+1} \quad \text{for } j \in [\ell], \text{for } k \in [n-1]$$

$$\mathcal{X}: \quad (w_{\hat{\imath},1} = w_{\hat{\imath},n}) \to (x_{i-1} \to x_i) \quad \text{for } i \in [m\ell]$$

where $\hat{\imath} = i \pmod{\ell}, x_0 := 1, x_{n\ell} := 0, m \simeq n, \ell \simeq \log n.$



Beyond Resolution

- How much should we focus on resolution anyway?
- Preprocessing $\Rightarrow$ introduce new variables $\Rightarrow$ extended resolution.

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- Modern solvers use inprocessing, this is now a pressing issue.
- Can still study DRAT without new variables as a proof system (DRAT^{-}).
- Many hard principles for resolution easy in DRAT^{-}.

[Buss, Thapen'19]

Take Home

- CDCL equivalent to Resolution
- But only under assumptions, not all reasonable

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Open Problems

- How to model space?
- Are restarts important?
- How much overhead do we need?

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Images: Vecteezy.com

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Thanks!