Size-Space Bounds and Trade-offs for CDCL Proofs

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FLoC Workshop on Proof Complexity

Algorithms for SAT [1960s]

- State-of-the-art 1960s: DPLL

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while not solved :
  unit_propagate()
  if conflict :


    backtrack()
  else :
    decide_variable_assignment()
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Algorithms for SAT

- State-of-the-art 1960s: DPLL
- State-of-the-art now: Conflict Driven Clause Learning

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while not solved :
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- Execution trace of CDCL $\rightarrow$ proof in subsystem of resolution
- Even true for most forms of preprocessing
- How strong a subsystem?
  - DPLL only as strong as tree-like
  - CDCL DAG-like, how much?

Known results

Line of research:

CDCL polynomially simulates resolution, but artificial models

[Beame, Kautz, Sabharwal '04], [Van Gelder '05],

[Hertel, Bacchus, Pitassi, Van Gelder '08], [Buss, Hoffmann, Johannsen '08]

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Under natural model (still some technical assumptions)

- CDCL p-simulates resolution [Pipatsrisawat, Darwiche '09]
- Randomized CDCL efficiently finds narrow proofs [Atserias, Fichte, Thurley '09]

Both results can be obtained from both papers

An even more faithful model?

Technical assumptions

- Unlimited memory
- Very frequent restarts
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Relevant questions

- Can we model memory/space?
- How important are restarts?
- Performance of actual heuristic vs random decisions?

An even more refined model

- Based on [Pipatsrisawat, Darwiche '09], [Atserias, Fichte, Thurley '09]; also ideas from [Buss, Hoffmann, Johannsen '08]

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- Proof as explicit resolution DAG + verification of CDCL-like

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- Proof as explicit resolution DAG + verification of CDCL-like
- Natural measure of space captures erasure of learnt clauses

A closer look at CDCL

```
while not solved :
  unit_propagate()     // Unit resolution
  if conflict :
    learn()
    maybe_restart()
    backjump()
  else :
    assign_variable() // New decision
```

Data structures

- Branching sequence (assignments by decision or propagation)
- Clause database (list of learned clauses)

A closer look at CDCL

(Partial) input: $z \wedge (x \vee y) \wedge (\overline{u} \vee \overline{w}) \wedge (u \vee \overline{w} \vee \overline{x} \vee y \vee \overline{z})$

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$z = 1 \quad z$

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$u \vee \overline{w} \vee \overline{x} \vee y \vee \overline{z}$

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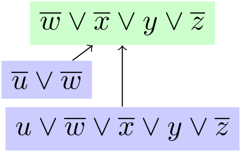

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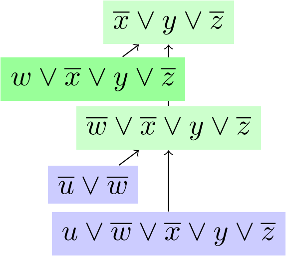

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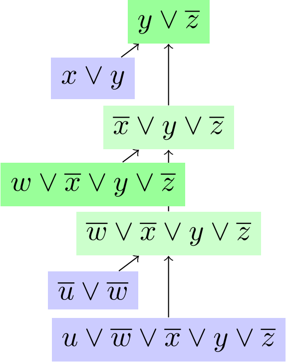

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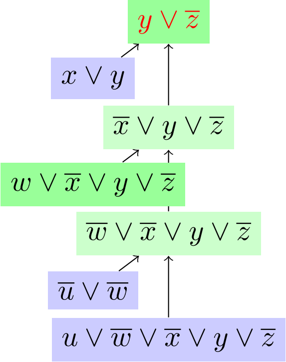

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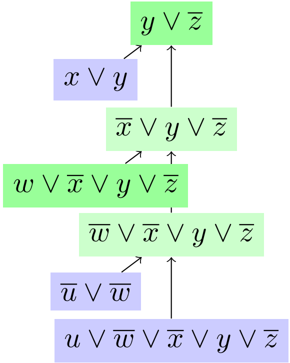
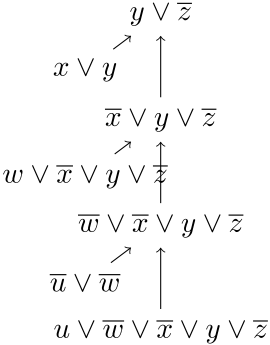


Next: $y = 1$ because of learned clause $y \vee \overline{z}$
Guaranteed by standard learning heuristic 1UIP

Proof system as annotated resolution

- Clauses as resolution DAG



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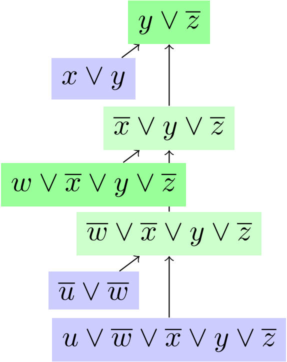

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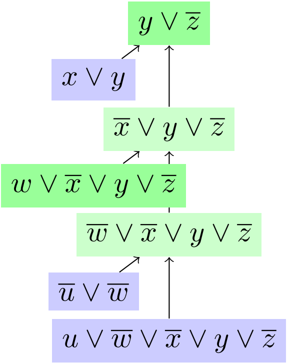
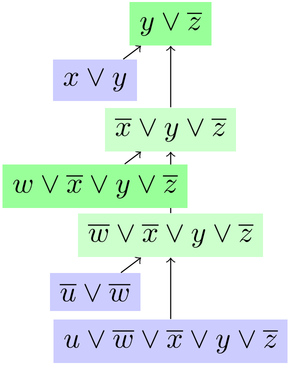


Proof system as annotated resolution

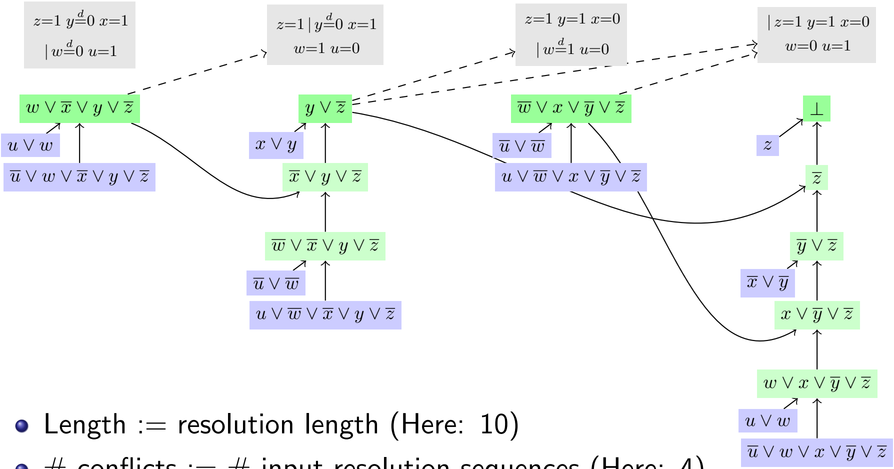
$z{=}1 \mid y \overset{d}{=} 0 \; x{=}1$

$w{=}1 \; u{=}0$

- Clauses as resolution DAG
- Grouped by sequences of input resolution
- Learned clauses allowed in later steps
- Branching sequence allows local checks



Standard measures



- Length := resolution length (Here: 10)
- # conflicts := # input resolution sequences (Here: 4)
- Space := database size (Here: 2)

Some facts

Theorem

CDCL proof system polynomially simulates resolution length

By [Pipatsrisawat, Darwiche '09], [Atserias, Fichte, Thurley '09]

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Observation

CDCL proofs are valid resolution proofs

Hence all lower bounds on length, space and trade-offs apply

Length and space upper bounds

Worst case for CDCL proof system same as resolution

Proposition

Every formula has proofs in length $O(2^n)$ and space $O(n)$ simultaneously

Not surprising, but also not immediate

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But can we simulate general resolution with respect to both length and space?

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Work in progress

Length-space trade-offs with restarts

Trade-offs from [Ben-Sasson, Nordström '11] also hold with

Theorem

There exists a family of formulas such that:

1. *There are short CDCL proofs*
2. *There are small CDCL proofs*
3. *Optimizing one measure blows up the other*

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Proof sketch.

(3) immediate from resolution [BN'11]
Are there matching proofs in CDCL?

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Are there matching proofs in CDCL? Yes

Simulate resolution clause by clause, restart at every clause

Space preserved

Standard 1UIP learning heuristic □

Trade-offs without restarts

Theorem

There exists a specific family of formulas such that:

1. *There are CDCL proofs in space* $s = O(1)$
2. *There are CDCL proofs in length* $L = O(n^2/s)$
3. *Every proof requires length* $L = \Omega(n^2/s^2)$

Line of research investigating power of restarts

Upper bounds rely on restarts. Necessary?

Trade-offs without restarts

Theorem

There exists a specific family of formulas such that:

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We craft explicit proofs without restarts for some families

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Plausible for most results from [Ben-Sasson, Nordström '11] to follow
Technically involved, work in progress

More trade-offs?

Open: analogous results for the trade-offs in [Beame, Beck, Impagliazzo '12] and [Beck, Nordström, Tang '13]

- Conceivable for CDCL
- Less clear without restarts

Summary

- New CDCL proof system faithfully models:
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  - Restarts
  - Learning heuristics

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